BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 80, Number 4, July 1974

ON VITALI-HAHN-SAKS TYPE THEOREMS

BY BARBARA FAIRES[1]

Communicated by Robert Bartle, November 7, 1973

In recent years extensive work has been done on the Vitali-Hahn-Saks theorem and its relatives. Seever **[13]** considered the question of extending the Vitali-Hahn-Saks theorem to the case where the domain is a Boolean algebra which is not necessarily sigma complete. Brooks and Jewett **[2]** established results for a strongly bounded map defined on a Boolean sigma algebra of sets with values in a Banach space. Further generalizations to group-valued set functions have been studied by the Poznán school (see **[5]**, **[6]**, **[7]**, **[8]**, **[9]**, **[11]**, **[12]**). The work of all these authors is generalized herein to the case of strongly bounded maps defined on Boolean algebras with the Seever property and taking values in a Banach space. Some applications other than those considered herein and the final generalization to the group-valued case can be found in **[10]**.

I wish to thank Professor J. Diestel for his advice and encouragement in the preparation of this paper. Also, I wish to express my gratitude to Professors R. E. Huff and J. J. Uhl, Jr. for many helpful discussions.

1. **Notation and definitions.** A Boolean algebra $\mathscr{B}$ has the *property* (I) if and only if for any sequences $\{x_n\}$ and $\{y_n\}$ in $\mathscr{B}$ satisfying $x_n \leqq y_m$ for all n, m, there exists $x \in \mathscr{B}$ such that $x_n \leqq x \leqq y_n$ for all n. This condition is equivalent to the condition: given any sequences $\{a_n\}$ and $\{b_n\}$ in $\mathscr{B}$ satisfying $a_n \wedge a_m = 0$, $b_n \wedge b_m = 0$ for $n \neq m$ and $a_n \wedge b_m = 0$ for all n, m, there exists an element a in $\mathscr{B}$ such that $a \geqq a_n$ and $b_n \wedge a = 0$ for all n.

Unless signified otherwise, $\mathscr{B}$ will be used in this paper to denote a Boolean algebra with the property (I). The symbol X denotes a Banach space and X^* its Banach space dual.

A finitely additive $\mu : \mathscr{B} \to X$ is *bounded* whenever there exists $M > 0$ such that $\|\mu(b)\| \leqq M$ for all $b \in \mathscr{B}$; μ is said to be *strongly bounded* if $\|\mu(e_n)\| \to 0$ as $n \to \infty$ for each disjoint sequence $e_1, \cdots, e_n, \cdots$ of elements in $\mathscr{B}$. A sequence $\mu_n : \mathscr{B} \to X$, $n = 1, 2, \cdots$, is *uniformly strongly bounded* if for each disjoint sequence $\{e_n\} \subset \mathscr{B}$, $\lim_n \sup_k \|\mu_k(e_n)\| = 0$. By grouping it is easy to see that if μ is strongly bounded and $\{e_n\} \subset \mathscr{B}$ is disjoint, then $\sum_{n=1}^{\infty} \mu(e_n)$ is an unconditionally convergent series in X. A map $\mu : \mathscr{B} \to X$ is

AMS (MOS) subject classifications (1970). Primary 46G99; Secondary 28A60.

[1] Supported in part by a Doctoral Fellowship granted by Kent State University.

countably additive if for every disjoint sequence $\{e_n\} \subset \mathscr{B}$ with $\bigvee_n e_n \in \mathscr{B}$, the equality $\mu(\bigvee_n e_n) = \sum_n \mu(e_n)$ holds. The *semivariation* of μ on $b \in \mathscr{B}$, denoted by $\|\mu\|(b)$, is defined to be $\sup\{\|\mu(a)\| : a \in \mathscr{B}, a \leqq b\}$. It is easily shown that $\mu : \mathscr{B} \to X$ is strongly bounded if and only if $\|\mu\| : \mathscr{B} \to [0, \infty)$ is strongly bounded (though $\|\mu\|$ need not be additive).

2. Main results.

THEOREM 1. *Let $\mu_n : \mathscr{B} \to X$ be finitely additive and strongly bounded for $n = 1, 2, \cdots$. If $\lim_n \mu_n(e) = 0$ for each $e \in \mathscr{B}$, then $\{\mu_n : n \in N\}$ is uniformly strongly bounded.*

PROOF. Suppose not. Then there exists a sequence $\{e_n\}$ of disjoint elements of $\mathscr{B}$, a number $\varepsilon > 0$, and a sequence $m_1 < m_2 < m_3 < \cdots$ of positive integers (to simplify notation, assume $m_n = n$) such that for each $n \in N$, $\|\mu_n(e_n)\| > 4\varepsilon$.

Let $i_1 = 1$. Partition the set $N \backslash \{1\}$ into an infinite number of infinite disjoint sets π_n^1, $n = 1, 2, 3, \cdots$. Utilizing property (I) we can choose a sequence f_n^1, $n = 1, 2, \cdots$, of disjoint elements in $\mathscr{B}$ such that:

(a_1) $f_n^1 \geqq e_i$ for all $i \in \pi_n^1$, $n = 1, 2, \cdots$;

(b_1) $f_n^1 \wedge e_{i_1} = 0$, $n = 1, 2, \cdots$;

(c_1) $f_n^1 \wedge e_j = 0$ for all $j \in (N \backslash \{1\}) \backslash (\bigcup_{i=1}^n \pi_i^1)$.

As $\|\mu_{i_1}\|(f_n^1) \to 0$ $(n \to \infty)$ there exists an $n_1 \in N$ such that $\|\mu_{i_1}\|(f_{n_1}^1) < \varepsilon$. Choose $i_2 \in \pi_{n_1}^1$ such that $i_2 > i_1$ and $\|\mu_{i_2}(e_{i_1})\| < \varepsilon/4$. Partition the set $\pi_{n_1}^1 \backslash \{i_2\}$ into an infinite number of infinite disjoint sets π_n^2, $n = 1, 2, \cdots$. Again by property (I) there exists a sequence f_n^2, $n = 1, 2, \cdots$, of disjoint elements in $\mathscr{B}$ such that

(a_2) $f_n^2 \geqq e_i$ for all $i \in \pi_n^2$, $n = 1, 2, \cdots$;

(b_2) $f_n^2 \wedge (e_{i_1} \vee e_{i_2}) = 0$, $n = 1, 2, \cdots$;

(c_2) $f_n^2 \wedge e_j = 0$ for all $j \in (\pi_{n_1}^1 \backslash \{i_2\}) \backslash (\bigcup_{i=1}^n \pi_i^2)$.

There exists an integer $n_2 \in N$ such that $\|\mu_{i_2}\|(f_{n_2}^2) < \varepsilon$. Choose $i_3 \in \pi_{n_2}^2$ such that $i_3 > i_2$ and $\|\mu_{i_3}(e_{i_1})\|$, $\|\mu_{i_3}(e_{i_2})\| < \varepsilon/8$. Proceed in this fashion to obtain a sequence $f_{n_k}^k = f_k$, $k = 1, 2, \cdots$, of elements of $\mathscr{B}$ and a sequence $i_1 < i_2 < \cdots$ of positive integers such that:

(1) $f_n \geqq e_{i_k}$, $k > n$;

(2) $f_n \wedge e_{i_k} = 0$, $1 \leqq k \leqq n$;

(3) $\|\mu_{i_n}\|(f_n) < \varepsilon$, $n = 1, 2, \cdots$;

(4) $\|\mu_{i_n}(e_{i_k})\| < \varepsilon/2^n$, $1 \leqq k < n$;

(5) $\|\mu_{i_n}(e_{i_n})\| > 4\varepsilon$, $n = 1, 2, \cdots$.

Let $h_n = f_n \vee (\bigvee_{k=1}^n e_{i_k})$. Then $h_n \geqq e_{i_k}$ for all n, k. Choose $c \in \mathscr{B}$ such that

$$h_n \geqq c \geqq e_{i_n} \quad \text{for all } n. \tag{6}$$

Noticing that $\mu_{i_n}(c)=\mu_{i_n}(h_n-e_{i_n})-\mu_{i_n}(h_n-c)+\mu_{i_n}(e_{i_n})$, we have

$$\begin{aligned}\|\mu_{i_n}(c)\| &\geqq \|\mu_{i_n}(e_{i_n})\| - \|\mu_{i_n}(h_n - e_{i_n})\| - \|\mu_{i_n}(h_n - c)\| \\ &= \|\mu_{i_n}(e_{i_n})\| - \left\|\mu_{i_n}\left[\left(f_n \vee \left(\bigvee_{k=1}^{n} e_{i_k}\right)\right) \wedge e'_{i_n}\right]\right\| \\ &\quad - \left\|\mu_{i_n}\left[\left(f_n \vee \left(\bigvee_{k=1}^{n} e_{i_k}\right)\right) \wedge c'\right]\right\|,\end{aligned}$$

which by (2) is

$$\begin{aligned}&\geqq \|\mu_{i_n}(e_{i_n})\| - \|\mu_{i_n}(f_n \wedge e'_{i_n})\| \\ &\quad - \left\|\mu_{i_n}\left[\bigvee_{k=1}^{n}(e_{i_k} \wedge e'_{i_n})\right]\right\| - \|\mu_{i_n}(f_n \wedge c')\| - \left\|\mu_{i_n}\left[\bigvee_{k=1}^{n}(e_{i_k} \wedge c')\right]\right\|.\end{aligned}$$

Applying (2), (6) and the disjointness of the e_{i_k}'s yields

$$\begin{aligned}&\geqq \|\mu_{i_n}(e_{i_n})\| - \|\mu_{i_n}(f_n)\| \\ &\quad - \|\mu_{i_n}(e_{i_n})\| - \cdots - \|\mu_{i_n}(e_{i_{n-1}})\| - \|\mu_{i_n}(f_n \wedge c')\|,\end{aligned}$$

which by (5), (3) and (4) is $>4\varepsilon-\varepsilon-(n-1)\varepsilon/2^n-\varepsilon\geqq\varepsilon$. Since $\|\mu_{i_n}(c)\|>\varepsilon$ holds for infinitely many n, $\lim_n \mu_n(c)\not\to 0$, a contradiction.

The proofs of some of the corollaries yielded by Theorem 1 are, for the most part, minor alterations to proofs presented elsewhere; in these cases the appropriate references are given.

COROLLARY 1 [2, COROLLARY 1.2]. *Let $\mu_n:\mathscr{B}\to X$ be finitely additive and strongly bounded for $n=1, 2, \cdots$. If $\lim_n \mu_n(e)=\mu(e)$ exists for each $e \in \mathscr{B}$, then μ is strongly bounded and the μ_n, $n=1, 2, \cdots$, are uniformly strongly bounded.*

COROLLARY 2. *Let $\mu_n:\mathscr{B}\to X$ be countably additive for $n=1, 2, \cdots$. If $\lim_n \mu_n(e)=\mu(e)$ exists for each $e \in \mathscr{B}$, then μ is countably additive and the μ_n, $n=1, 2, \cdots$, are uniformly countably additive.*

COROLLARY 3 [3, THEOREM 1.6]. *Let X be any separable Banach space and let $\mu:\mathscr{B}\to X$ be bounded and finitely additive. Then μ is strongly bounded.*

Another corollary is the following result proved differently by N. J. Kalton in an unpublished manuscript.

COROLLARY 4. *Let X be a weakly compactly generated Banach space and let $\mu:\mathscr{B}\to X$ be bounded and finitely additive. Then μ is strongly bounded.*

PROOF. Let $\{e_n\}$ be a disjoint sequence in $\mathscr{B}$ and let $[\mu(e_n)]=X_0$ denote the closed linear span of $\{\mu(e_n):n \in N\}$. Then X_0 is a separable subspace of the weakly compactly generated space X; hence by a result of Amir and Lindenstrauss [**1**, Lemma 4], there is a separable subspace Y of X such that $X_0 \subset Y$ and Y is complemented in X. Suppose $P:X\to Y$ is the projection. Then Corollary 1.2 yields $P \circ \mu(e_n)\to 0$, $n\to\infty$. But $P \circ \mu(e_n)=\mu(e_n)$ for each n. Therefore, μ is strongly bounded.

COROLLARY 5 [**4**, COROLLARY 5]. *Let $\mu_n:\mathscr{B}\to X$ be strongly bounded for $n=1, 2, \cdots$. Suppose $\mu(e)=$ weak-limit$_n\, \mu_n(e)$ exists for each $e \in \mathscr{B}$. Then μ is strongly bounded.*

PROOF. The boundedness of μ follows from the Banach-Steinhaus theorem and Corollary 1.1 applied to the functions $f\mu_n$, $f\mu$ where $f\in X^*$. For each n let $B_n=\mu_n(\mathscr{B})$ and let Y be the closed linear span of $\bigcup_n B_n$. By the definition of μ and Mazur's theorem we have $\mu(\mathscr{B})\subset Y$. We claim that Y is weakly compactly generated.

For each n, let $M_n=\sup\{\|\mu_n(b)\|:b \in \mathscr{B}\}$. Let $B=\bigcup_n B_n/(n \cdot M_n)$. The closed linear span of B is Y and B is relatively weakly compact. To see the last assertion, let $\{y_n\}$ be a sequence in B. Since each μ_n is strongly bounded, B_n, and hence $B_n/(n \cdot M_n)$, is relatively weakly compact [**14**]. So if $\{y_n\}$ returns infinitely often to one of the $B_n/(n \cdot M_n)$'s, we can extract a weakly convergent subsequence. If $\{y_n\}$ does not return infinitely often to any $B_n/(n \cdot M_n)$ then there exist strictly increasing sequences (m_k) and (n_k) of positive integers such that $y_{m_k} \in B_{n_k}/(n_k \cdot M_{n_k})$ for each k. It follows that $\|y_{m_k}\|\leqq 1/n_k\to 0$ as $k\to\infty$. Thus $\{y_n\}$ has a norm convergent subsequence.

With the proof proceeding as in [**2**, Theorem 3] we have the following Vitali-Hahn-Saks theorem.

THEOREM 2. *Let $\mu_n:\mathscr{B}\to X$ be finitely additive and strongly bounded, for $n=1, 2, \cdots$. Suppose ν is a nonnegative monotone set function defined on $\mathscr{B}$ and each $\mu_n\ll\nu$. Assume that* $\lim_n \mu_n(e)$ *exists for each $e \in \mathscr{B}$. Then* $\lim_{\nu(e)\to 0}\|\mu_n(e)\|=0$ *uniformly in n.*

REFERENCES

1. D. Amir and J. Lindenstrauss, *The structure of weakly compact sets in Banach spaces*, Ann. of Math. (2) **88** (1968), 35–46. MR **37** #4562.

2. J. K. Brooks and R. S. Jewett, *On finitely additive vector measures*, Proc. Nat. Acad. Sci. U.S.A. **67** (1970), 1294–1298. MR **42** #4697.

3. J. Diestel, *Applications of weak compactness and bases to vector measures and vectorial integration*, Rev. Roumaine Math. **18** (1972), 211–224.

4. ———, *Grothendieck spaces and vector measures*, Vector and Operator Valued Measures and Applications, Academic Press, New York, 1973.

5. L. Drewnowski, *Control submeasures and control measures*, Studia Math. (to appear).

6. ———, *Decompositions of set functions*, Studia Math. **48** (1973), 23–48.

7. ———, *Topological rings of sets, continuous set functions, integration.* I, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **20** (1972), 269–276. (Russian) MR **46** #5558.

8. ———, *Topological rings of sets, continuous set functions, integration.* II, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **20** (1972), 277–286. (Russian) MR **46** #5558.

9. ———, *Topological rings of sets, continuous set functions, integration.* III, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **20** (1972), 439–445.

10. Barbara Faires, *Grothendieck spaces and vector measures*, Ph.D. Dissertation, Kent State University, Kent, Ohio, August, 1974.

11. I. Labuda, *On some generalizations of the theorems of Nikodym and of Vitali-Hahn-Saks*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **20** (1972), 447–456.

12. ———, *Sur le Théorème de Bartle-Dunford-Schwartz*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **20** (1972), 549–553.

13. G. L. Seever, *Measures on F-spaces*, Trans. Amer. Math. Soc. **133** (1968), 267–280. MR **37** #1976.

14. J. J. Uhl, Jr., *Extensions and decompositions of vector measures*, J. London Math. Soc. (2) **3** (1971), 672–676. MR **44** #4181.

Department of Mathematics, Kent State University, Kent, Ohio 44240