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ON THE BERGMAN KERNEL AND BIHOLOMORPHIC MAPPINGS OF PSEUDOCONVEX DOMAINS

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THEOREM 1. *Let* $D_1, D_2 \subset C^n$ *be strictly pseudoconvex domains with smooth boundaries and suppose that* $F: D_1 \to D_2$ *is biholomorphic* (*i.e.*, F *is an analytic homeomorphism*). *Then* F *extends to a diffeomorphism of the closures*, $\bar{F}: \bar{D}_1 \to \bar{D}_2$.

The main idea in proving Theorem 1 is to study the boundary behavior of geodesics in the Bergman metrics (see [2]) of D_1 and D_2. To do so, we use a rather explicit formula for the Bergman kernels of D_1 and D_2. We begin with a few definitions. Let $D = \{z \in C^n | \psi(z) > 0\}$ be a strictly pseudoconvex domain, where $\psi \in C^\infty(C^n)$ satisfies grad $\psi \neq 0$ on ∂D.

(1) Let $\mathscr{L}(\omega)$ denote the Levi form, i.e. the quadratic form

$$\mathscr{L}(\omega)\, dz\, \overline{dz} = \sum_{j,k} \frac{\partial^2(-\psi)}{\partial z_j\, \partial \bar{z}_k}\bigg|_\omega dz_j\, \overline{dz}_k$$

restricted to the subspace $\{dz \in C^n | \sum_j (\partial\psi/\partial z_j)|_w\, dz_j = 0\}$ of C^n.

(2) For $\omega_1, \omega_2 \in D$, set $\rho(\omega_1, \omega_2) = |\omega_1 - \omega_2|^2 + |(\omega_2 - \omega_1) \cdot (\partial\psi/\partial\omega)|_{\omega_1}|$. (See [2] again.)

(3) A smooth function φ defined on $\bar{D} \times \bar{D}$ has *weight* k (where $k \geqq 0$ is an integer or half-integer) if the following estimate holds.

$$|\varphi(\omega_1, \omega_2)| \leqq C(\psi(\omega_1) + \psi(\omega_2) + \rho(\omega_1, \omega_2))^k$$

(4) Set

$$X(z, \omega) = \psi(\omega) + \sum_j \frac{\partial\psi}{\partial\omega_j}\bigg|_\omega (z_j - \omega_j) + \frac{1}{2} \sum_{j,k} \frac{\partial^2\psi}{\partial\omega_j\, \partial\omega_k}\bigg|_\omega (z_j - \omega_j)(z_k - \omega_k).$$

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Elementary calculations show that $X(z, \omega)$ has weight 1, and that $|X(z, \omega)| \geqq c(\psi(z)+\psi(\omega)+\rho(z, \omega))$ in a region of the form $R_\delta = \{(z, \omega) \in \bar{D} \times \bar{D} \mid \psi(z)+\psi(\omega)+|z-\omega|<\delta\}$.

THEOREM 2. *The Bergman kernel $K(z, \omega)$ for D has an asymptotic expansion*

$$K(z, \omega) \sim c\,|\operatorname{grad} \psi(\omega)|^2 \det \mathscr{L}(\omega) X^{-(n+1)}(z, \omega) + \sum_{j=1}^{\infty} \varphi_j(z, \omega) X^{-m_j}(z, \omega) + \tilde{\varphi}(z, \omega) \log X(z, \omega), \tag{5}$$

where c is a constant, φ_j and $\tilde{\varphi}$ are smooth functions, "log" *denotes the principal branch of the logarithm on* $\{\operatorname{Re}(\zeta)>0\}$, *weight* $(\varphi_j)-m_j \geqq -n-\frac{1}{2}$, *and weight* $(\varphi_j)-m_j \to \infty$ *as* $j \to \infty$. *The expansion* (5) *is valid in a region* R_δ, *and the symbol* "$\sim$" *means that for any integer* k,

$$K(z, \omega) - c\,|\operatorname{grad} \psi(\omega)|^2 \det \mathscr{L}(\omega) X^{-(n+1)}(z, \omega) - \sum_{j=1}^{N} \varphi_j(z, \omega) X^{-m_j}(z, \omega) - \tilde{\varphi}(z, \omega) \log X(z, \omega) \in C^k(\bar{R}_\delta)$$

for N large enough.

COROLLARY. $K(z, z) = \Phi(z)\psi^{-(n+1)}(z) + \tilde{\Phi}(z) \log \psi(z)$, *where* $\Phi, \tilde{\Phi} \in C^\infty(\bar{D})$ *and* $\Phi \neq 0$ *near* ∂D.

Although $\tilde{\Phi}$ vanishes on the unit ball, it can be nonzero, even on very smooth (say, real-analytic) domains.

The proof of Theorem 2 is based on an elementary fact.

LEMMA 1. *Given $p \in \partial D$, we can find a region $\tilde{D}$ internally tangent to D to third order at p, and an explicit biholomorphic change of co-ordinates $\tilde{F}$ mapping a neighborhood of p in D to a neighborhood of $\tilde{F}(p)$ in the unit ball.*

Once Lemma 1 is established, we can use $\tilde{F}$ to pull the Bergman kernel from the unit ball back to $\tilde{D}$; and since $\tilde{D}$ so closely approximates D near p, we may hope that the (known) Bergman kernel for D provides a close approximation to the (unknown) Bergman kernel for $\tilde{D}$. Having thus obtained a candidate for an approximate Bergman kernel, we use a successive approximation procedure to prove (5).

Now we can attack Theorem 1 by using the corollary to Theorem 2 to make explicit differential-geometric calculations with the Bergman metric. We need two more definitions.

(6) For a fixed point $z^0 \in D$ and a unit vector $\omega \in S^{2n-1} \subseteq C^n$, let $t \to \gamma(t, \omega, z^0)$ be the path of a particle moving with unit speed (in the Bergman metric) along the geodesic in D starting at $t=0$ at the point

z^0 and travelling in the direction ω. We say that $(z^0, \omega^0) \in D \times S^{2n-1}$ is *pseudotransversal* if the map $\omega \to \pi_{z^0}(\omega) = \lim_{t\to\infty} \gamma(t,\omega,z^0)$ is well defined for ω close to ω^0 in S^{2n-1} and provides a diffeomorphism of a small open neighborhood of $\omega^0 \in S^{2n-1}$ onto a small open neighborhood of $\pi_{z^0}(\omega) \in \partial D$.

(7) Let $t \to \gamma(t)$ be a geodesic in D, and define $\omega_\gamma(t)$ = the unit vector in the direction $d\gamma(t)/dt$. If $(\gamma(t), \omega_\gamma(t)) \in D \times S^{2n-1}$ is pseudotransversal for all t larger than some fixed T, then we call γ a *pseudotransversal geodesic*.

LEMMA 2. (a) *Every geodesic $\gamma(t)$ not remaining in a fixed compact subset of D for all $t \geq 0$ is pseudotransversal.*

(b) *Every point $p \in \partial D$ is $\pi_{z^0}(\omega^0)$ for a certain $(z^0, \omega^0) \in D \times S^{2n-1}$.*

Theorem 1 is a simple consequence of Lemma 2 and a result of Vormoor [1] which states that under the hypotheses of Theorem 1, F extends to a continuous mapping $\bar{F}: \bar{D}_1 \to \bar{D}_2$. For, given $p_1 \in \partial D_1$, we use Lemma 2(b) to find a geodesic $\gamma_1(t)$ in D_1 with $\lim_{t\to\infty} \gamma_1(t) = p_1$. Since F is an isometry of Bergman metrics, the path $\gamma_2(t) = F(\gamma_1(t))$ is a geodesic in D_2, and by Lemma 2(a), both γ_1 and γ_2 are pseudotransversal. Set $p_2 = \lim_{t\to\infty} \gamma_2(t)$, and pick T so large that $(z_1, \omega_1) = (\gamma_1(T), \omega_{\gamma_1}(T))$ and $(z_2, \omega_2) = (\gamma_2(T), \omega_{\gamma_2}(T))$ are both pseudotransversal. Since the differential of F induces a diffeomorphism $(dF)^\sim$ between the unit tangent vectors based at z_1 and those based at z_2, we have a commutative diagram

$$\begin{tikzcd} S^{2n-1} \arrow[r, "(dF)^\sim"] \arrow[d, "\pi_{z_1}"'] & S^{2n-1} \arrow[d, "\pi_{z_2}"] \\ \partial D_1 \arrow[r, "\bar{F}"] & \partial D_2 \end{tikzcd}$$

where the maps π_{z_1} and π_{z_2} are defined in small neighborhoods of $\omega_1 = \pi_{z_1}^{-1}(p_1)$ and $\omega_2 = \pi_{z_2}^{-1}(p_2)$. All the maps in the diagram, except $\bar{F}$, are already known to be diffeomorphisms. Hence $\bar{F}$ must also be a diffeomorphism from a neighborhood of p_1 to a neighborhood of p_2, which proves Theorem 1.

REFERENCES

1. H. Grauert, Lecture given at a conference on several complex variables held in Paris, 1972.

2. E. M. Stein, *Boundary behavior of holomorphic functions of several complex variables*, Princeton Univ. Press, Princeton, N.J., 1972.

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