BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 80, Number 4, July 1974

MANIFOLDS WITH THE FIXED POINT PROPERTY. I

BY S. Y. HUSSEINI[1]

Communicated by Mary Ellen Rudin, November 6, 1973

1. **Introduction.** Suppose that $f: M \to M$ is a map of the simply connected closed (smooth or PL) manifold M which preserves a given geometric structure. We shall consider the question of when f has a fixed point. (The geometric structure is described by an element ξ in $K_R(M)$, the Grothendieck group of real vector bundles over M. If $\deg f = 1$, then for f to preserve ξ means just that $f^*\xi = \xi$, and the appropriate notion when $\deg f \neq 1$ is given below in §2. Such maps are said to be (ξ, λ)-maps with λ an integer.) Since M is simply connected, one need only compute the Lefschetz number $\mathscr{L}(f)$ of f. Thus there are three natural stages to the solution: the determination of the induced homomorphism $f^*: H^*(M; Z) \to H^*(M; Z)$ first below the middle dimension, then in the middle dimension (when dim M is even), and finally the determination of how the two are related to each other and how they determine the behaviour above the middle dimension.

As a first step in this direction, we consider here the case of $(2m-1)$-connected M of dimension $4m$ whose intersection pairing is definite (said to be of class $\mathscr{M}_{4m}$). It is shown that if ξ is asymmetric enough in a suitable sense (described below in §2), then any (ξ, λ)-map $f: M \to M$ has a fixed point. In particular it follows that if the tangent bundle $\tau(M)$ of M is asymmetric enough, then a $(\tau M, 1)$-map $f: M \to M$ has a fixed point. Therefore every homeomorphism of such a manifold M has a fixed point. It is also shown that the product of (ξ, λ)-maps with ξ being asymmetric also has a fixed point.

[*Note.* At this point I would like to thank Ed Fadell for the suggestions and stimulation offered in many good conversations on this topic.]

2. **Statement of results.** Suppose that M is a smooth (or PL) simply connected closed manifold of dimension $4m$. A map $f: M \to M$ is said to be a *(ξ, λ)-map*, where λ is an integer, if and only if $f^*\xi = \lambda\xi + p^*\eta$ where

AMS (MOS) subject classifications (1970). Primary 55C20, 57D99; Secondary 54H25, 15A63.

[1] While working on this paper the author has been partially supported by the National Science Foundation GP-29538-A4.

$p: M \to S^{4m}$ is a map of degree 1 and $\eta \in K_R(S^{4m})$. (Note that a diffeomorphism $f: M \to M$ is a $(\tau M, 1)$-map, $\tau(M)$ being the tangent bundle of M.)

Assume now that M is $(2m-1)$-connected, and suppose that the intersection pairing

$$\varphi: H^{2m}(M; Z) \times H^{2m}(H; Z) \to Z$$

is positive definite. The class of such manifolds will be denoted by $\mathcal{M}_{4m}$. (See [1] for their classification.) Let $c_m = c_m(\xi^c) \in H^{2m}(M; Z)$ be the Chern class of $\xi^c = \xi \otimes C$, the complexification of ξ, and assume that $c_m(\xi^c) \neq 0$. One can easily show that this implies that $\deg f = \lambda^2$ and the Lefschetz number $\mathcal{L}(f)$ is given by $\mathcal{L}(f) = 1 + s\lambda + \lambda^2$, with s rational and $|s| \leqq \sigma$, σ being the signature of φ. Hence $\mathcal{L}(f) \neq 0$ for $|\lambda| > \sigma$. On the other hand, the behaviour of $\mathcal{L}(f)$ for $|\lambda| \leqq \sigma$ is quite different, and thus σ is, in a sense, a critical threshold.

To describe the case $|\lambda| \leqq \sigma$, one shows first *that there is a basis* $\mathcal{S} = \{x_1, \cdots, x_\sigma\}$ *for* $H_{2m}(M; Z)$ *with the property that* $\langle x_i, c_m \rangle = \beta^{s_i}$ *where* $\beta = \min\langle x, c_m \rangle$ *and* s_i *are integers such that* $s_1 = 1$, $s_j - s_i > 0$ *for all* $j > i$, *and* c_m *the mth Chern class of* ξ.

The basis $\mathcal{S} = \{x_1, \cdots, x_\sigma\}$ defines a critical region for ξ. If $x, y \in H_{2m}(M; Z)$ and xy denotes their intersection number, then the critical region is the set

$$B_{\mathcal{S}} = \{x \in H_{2m}(M; Z) \mid x^2 \leqq \sigma^2 \mu_{\mathcal{S}}\}$$

where $\mu_{\mathcal{S}} = \max_i x_i^2$. Now let $\beta_{\mathcal{S}}$ be the smallest integer such that $|a_i| < \beta_{\mathcal{S}} - \sigma$ for all i, where $\sum_i a_i x_i \in B_{\mathcal{S}}$. ξ will be said to be *sufficiently asymmetric* if, and only if, $\beta \geqq \beta_{\mathcal{S}}$.

THEOREM 2.1. *Suppose that* ξ *is sufficiently asymmetric. Then any* (ξ, λ)-*map* $f: M \to M$ *has a fixed point, where* $M \in \mathcal{M}_{4m}$ *and* $m > 4$.

The following is an immediate consequence.

THEOREM 2.2. *Suppose that* $M \in \mathcal{M}_{4m}$ *with m even and* >4, *and assume that* $\tau(M)$, *the tangent bundle of* M, *is sufficiently asymmetric. Then any* $(\tau M, 1)$-*map* $f: M \to M$ *has a fixed point. In particular, any homeomorphism of M has a fixed point.*

The next theorem describes the behaviour of the products of (ξ, λ)-maps.

THEOREM 2.3. *Suppose that* M' *and* M'' *are two manifolds in* $\mathcal{M}_{4m'}$ *and* $\mathcal{M}_{4m''}$ *with* $m', m'' > 4$. *Let* $\xi' \in K_R(M')$ *and* $\xi'' \in K_R(M'')$ *be sufficiently asymmetric, and put* $\xi = \xi' \boxtimes \xi''$ *where* $\boxtimes$ *is the tensor product. Then any* (ξ, λ)-*map* $f: M' \times M'' \to M' \times M''$ *has a fixed point.*

3. **Construction of** (ξ, λ)**-maps.** In view of the preceding, it is important to know whether there is a (ξ, λ)-map $f: M \to M$. A map such as f has degree λ^2, and therefore the question becomes whether there is a map $f: M \to M$ of a given degree and whether a map of a given degree preserves a given $\xi \in K_{\mathbf{R}}(M)$. Let therefore $\alpha: H_{2m}(M; \mathbf{Z}) \to \pi_{2m-1}SO$ be the map which associates to x the characteristic class of the induced bundle $g^*\tau(M)$, g being an imbedding $S^{2m} \to M$ realizing x.

THEOREM 3.1. *Suppose that* $\gamma: H^{2m}(M; \mathbf{Z}) \to H^{2m}(M; \mathbf{Z})$ *is a monomorphism such that* $\varphi(\gamma x, \gamma y) = \lambda^2 \varphi(x, y)$ *for all* $x, y \in H^{2m}(M; \mathbf{Z})$, *where* λ *is a given integer and* φ *is the intersection pairing in* M. *Assume also that* $\gamma(\alpha) = \lambda\alpha$. *Then there is a map* $f: M \to M$ *such that* γ *is the induced homomorphism on cohomology, provided that* $J(\lambda(\lambda-1)\alpha(x)) = 0$ *for all* $x \in H_{2m}(M; \mathbf{Z})$, *with* J *being the* J*-homomorphism* (*cf.* [**2**, *Lemma* 10] *and* [**1**, *Theorem* 5]).

Whether or not a map $f: M \to M$ of a given degree preserves a given $\xi \in K_{\mathbf{R}}(M)$ is decided by considering the characteristic classes of ξ and $f^*\xi$.

Thus the question of finding a (ξ, λ)-map $f: M \to M$ amounts to finding a homomorphism $\gamma: H^{2m}(M; \mathbf{Z}) \to H^{2m}(M; \mathbf{Z})$ which preserves the intersection pairing φ, the stable tangential structure α, and the Chern class of ξ. If M is almost parallelizable, then α is trivial, $\tau(M)$ has a large measure of symmetry, and the existence of (ξ, λ)-maps depends only on ξ and how large the group of automorphisms of φ is. In particular, if $\lambda = 1$ and $\xi = \tau(M)$, it follows that every quadratic automorphism $\gamma: H^{2m}(M; \mathbf{Z}) \to H^{2m}(M; \mathbf{Z})$ is induced by a corresponding homeomorphism $f: M \to M$.

REFERENCES

1. C. T. C. Wall, *Classification of* $(n-1)$*-connected* $2n$*-manifolds*, Ann of Math. (2) **75** (1962), 163–189. MR **26** #3071.

2. ———, *Classification problems in differential topology*. II. *Diffeomorphisms of handlebodies*, Topology **2** (1963), 263–272. MR **27** #6278.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF WISCONSIN, MADISON, WISCONSIN 53706