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STABLE MINIMAL SURFACES

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Let $M \subset R^3$ be a minimal surface. A *domain* $D \subset M$ is an open connected set with compact closure $\bar{D}$ and such that its boundary ∂D is a finite union of piecewise smooth curves. We say that D is *stable* if D is a minimum for the area function of the induced metric, for all variations of $\bar{D}$ which keep ∂D fixed. In this note we announce the following estimate of the "size" of a stable minimal surface. We will denote by S^2 the unit sphere of R^3.

THEOREM. *Let $g: M \subset R^3 \to S^2$ be the Gauss map of a minimal surface M and let $D \subset M$ be a domain. If* area $g(D) < 2\pi$ *then D is stable.*

REMARK. The estimate is sharp, as can be shown, for instance, by considering pieces of the catenoid bounded by circles C_1 and C_2 parallel to and in opposite sides of the waist circle C_0. By choosing C_1 close to C_0 and C_2 far from C_0, we may obtain examples of unstable domains whose area of the spherical image is bigger than 2π and as close to 2π as we wish.

REMARK. The theorem implies that if the total curvature of D is smaller than 2π, then D is stable. The theorem is however stronger since we only use the area of the spherical image and the total curvature is equal to this area counting multiplicity.

REMARK. Our theorem is related to a result of A. H. Schwarz (see, for instance, [3, p. 39]).

We want to thank S. S. Chern for having suggested this question to us. Thanks are also due to R. Osserman, who read critically a preliminary version of this work, and to R. Gulliver, who pointed out some gaps in our first proof.

Below we present a brief sketch of the proof. A complete proof along with further results will appear elsewhere.

SKETCH OF THE PROOF. Let Δ and K be the laplacian and the gaussian curvature of M, respectively, in the induced metric. Assume that D is not stable. It follows from the Morse index theorem [4] that there

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exists a domain $D' \subset D$ and a C^∞-function u on D' such that $u>0$ on D', $u\equiv 0$ on $\partial D'$, and $\Delta u - 2uK = 0$. Since g is antiholomorphic and $\bar{D}'$ is compact, there are only finitely many points $p_1^0, \cdots, p_m^0$ where $K=0$. Thus, in $D' - \bigcup \{p_j^0\}$, $j=1, \cdots, m$, g is a local diffeomorphism, and for each $q \in g(\bar{D}')$, $g^{-1}(q) \cap \bar{D}^1 = \{p_1, \cdots, p_n\}$ is finite. We then define a function f on $g(\bar{D}')$ by

$$f(q) = \sum_i u(p_i), \qquad i = 1, \cdots, n.$$

It is easily seen that f is not identically zero and $f\equiv 0$ on $\partial(g(D'))$. Furthermore, it can be proved that f is continuous in $g(\bar{D}')$ and differentiable in $g(\bar{D}') - g(\partial D' \cup \{p_1^0, \cdots, p_m^0\})$.

A crucial point in the proof is to show that

$$(1) \qquad \int_{g(\bar{D}')} |\mathrm{grad}\, f|^2 \, dS \leqq 2 \int_{g(\bar{D}')} f^2 \, dS,$$

where dS is the element of area of S^2. This is accomplished by decomposing $g(\bar{D}')$ in a suitable way and, using the fact that g is antiholomorphic, by lifting parts of the above integrals into $\bar{D}'$, where they can be more easily computed. The remaining parts are then estimated and this yields the required inequality.

From now on, it will be convenient to denote by λ_1^T the first eigenvalue for the spherical laplacian of a domain $T \subset S^2$. It follows from (1) that $\lambda_1^{g(D')} \leqq 2$.

A further important point in the proof is to show that, given a domain $T \subsetneqq S^2$, it is possible to deform it, keeping its area fixed and not increasing its first eigenvalue, in such a way that it will eventually lie inside an open "cap" (i.e., a domain in S^2 bounded by an equator). To do this, we adapt, for functions and domains on a sphere, the process of circular symmetrization about an axis known in the plane (see [**2**, p. 193]). Using the ideas of [**2**] it is not difficult to show that such a process keeps the area of T fixed and does not increase λ_1^T. The main point is now to prove that by successive applications of this process, the boundary of T converges to a parallel of the sphere. This requires estimating some geometrical quantities which appear in the process of symmetrization.

It follows that $g(D')$ can be symmetrized into a domain $g(D')^*$, such that $g(D')^*$ is properly contained in a hemisphere $H \subset S^2$ and $\lambda_1^{g(D')^*} \leqq \lambda_1^{g(D')} \leqq 2$. Since $\lambda_1^H = 2$, and a proper inclusion increases strictly the first eigenvalue, we obtain that $\lambda_1^{g(D)^*} > 2$, and this contradiction proves the theorem.

REMARK. The above process of symmetrization can be used to give a proof of the isoperimetric inequality on the sphere. Our convergence argument also contains a proof of a spherical version of the Faber-Krahn inequality (see [**1**, p. 413]).

REMARK. The arguments which lead to inequality (1) can also be used to prove that if $g(D)\subset T\subset S^2$ and $\lambda_1^T>2$ then D is stable.

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