BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 80, Number 3, May 1974

ON QUOTIENTS OF MANIFOLDS: A GENERALIZATION OF THE CLOSED SUBGROUP THEOREM

BY HÉCTOR J. SUSSMANN[1]

Communicated by Glen E. Bredon, October 10, 1973

Let M be a (C^∞, Hausdorff, paracompact) manifold, and let R be an equivalence relation on M. Then R is called *regular* if the quotient M/R is a (not necessarily Hausdorff) manifold in such a way that the canonical projection $\pi_R: M \to M/R$ is a submersion. For results on regular relations cf. Palais [**1**], Serre [**2**]. The following characterization of regularity is well-known (cf. Serre [**2**, LG, Chapter 3, §12]): R is regular if and only if it is a submanifold (with the subspace topology) of $M \times M$ in such a way that the map $(m, m') \to m$ from R onto M is a submersion.

The purpose of this note is to announce a different characterization of regularity. Proofs will appear elsewhere (Sussmann [**3**]).

Our condition is motivated in a natural way by Systems Theory. As will be shown in [**4**], Theorem 2 is precisely what is needed to show that, under fairly general conditions, every finite-dimensional "controllable" nonlinear system has a realization which is both "controllable" and observable.

Here we shall not pursue this line. Rather, we shall state our condition and show that it is a rather natural generalization of the closed subgroup theorem.

Let X be a vector field on an open subset of M. We say that X is a *symmetry vector field of* R if, whenever $(m, m') \in R$, it follows that $(X_t(m), X_t(m')) \in R$ for every real t for which $X_t(m)$ and $X_t(m')$ are both defined (here $t \to X_t(m)$ is the integral curve of X which passes through m when $t=0$). Let $S^\infty(R, M)$ denote the set of all C^∞ vector fields X defined on open subsets of M that are symmetry vector fields for R. It is not difficult to show that $S^\infty(R, M)$ is a presheaf of Lie algebras of vector fields. If L is a set of vector fields defined on open subsets of M, we say that L is *transitive* if, for every $m \in M$, the vectors $X(m)$, $X \in L$, span the tangent space of M at m. If A is a subset of $M \times M$, we call L *A-transitive* if, for every $(m, m') \in A$, the tangent space of M at m is spanned by the

AMS (MOS) subject classifications (1970). Primary 58A05.

[1] Work partially supported by NSF Grant No. GP-37488.

vectors $X(m)$, where $X \in L$ is a vector field whose domain of definition contains both m and m'.

The equivalence relation R is called *locally regular* if every $m \in M$ has a neighbourhood U such that $R \cap (U \times U)$ is a regular equivalence relation in U. Similarly, we call R locally closed if every $m \in M$ has a neighborhood U such that $R \cap (U \times U)$ is closed in $U \times U$.

THEOREM 1. *R is locally regular if and only if it is locally closed and $S^\infty(R, M)$ is transitive.*

THEOREM 2. *Let M be connected. Then R is regular if and only if it is locally closed and $S^\infty(R, M)$ is R-transitive.*

REMARK. As is well known, if R is regular, then M/R is Hausdorff if and only if R is closed.

When M is not connected, the condition of Theorem 2 is sufficient for regularity, except for the fact that the connected components of M/R will not necessarily have the same dimension. Let us call R *almost regular* if every connected component M' of M/R can be given a differentiable structure in such a way that the canonical projection from $\pi_R^{-1}(M')$ onto M' is a submersion. Then Theorem 2 has an analogue that does not require M to be connected.

THEOREM 2′. *Let M be an arbitrary C^∞ manifold. Then R is almost regular if and only if it is locally closed and $S^\infty(R, M)$ is R-transitive.*

The preceding theorems have been stated in the C^∞ category, but they are equally valid in the real analytic case (for instance, if M is a connected real analytic manifold, and if the set $S^\omega(R, M)$ of real analytic symmetry vector fields of R is R-transitive, then M/R is a real analytic manifold in such a way that $\pi_R: M \rightarrow M/R$ is a real analytic submersion). The following two theorems, however, are stated for the C^∞ case, and our proofs depend on arguments that have no real analytic analogue (such as the use of C^∞ functions with compact support). We do not know whether the "only if" parts of Theorems 3 and 4 are also valid in the real analytic case (but it is easy to see that the "if" parts are).

THEOREM 3. *Let M be a connected C^∞ manifold. Then the equivalence relation R is regular if and only if R is locally closed and the set of everywhere defined symmetry vector fields of R is transitive.*

A Theorem 3′ (in which M is not required to be connected and regularity is replaced by almost regularity) is also valid.

Finally, the following characterizes those relations R for which π_R is a fibre map.

THEOREM 4. *Let M be a connected C^∞ manifold and let R be an equivalence relation on M. Then R is regular and the projection π_R is a fibre map, if and only if R is locally closed and there exists a transitive set of complete everywhere defined symmetry vector fields of R.*

Again, there is an analogous theorem for the nonconnected case, in which regularity is replaced by almost regularity.

We remark that Theorem 4 can be applied in particular if M is a Lie group and if R is the equivalence relation induced by a closed subgroup H. The transitive set of symmetry vector fields is, simply, the Lie algebra of M. Therefore Theorem 4 implies that R is regular (so that H is a regular submanifold of M) and that G/H is a manifold in such a way that the projection $g \to gH$ is a fibre map. This shows that the closed subgroup theorem is contained on our results.

REMARK. Another particular case of Theorem 4 is Ehresman's theorem on proper submersions.

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DEPARTMENT OF MATHEMATICS, RUTGERS UNIVERSITY, NEW BRUNSWICK, NEW JERSEY 08903