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AN INEQUALITY FOR THE DISTRIBUTION OF A SUM OF CERTAIN BANACH SPACE VALUED RANDOM VARIABLES

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1. **Introduction.** Throughout the paper B is a real separable Banach space with norm $\|\cdot\|$, and all measures on B are assumed to be defined on the Borel subsets of B. We denote the topological dual of B by B^*.

A measure μ on B is called a mean zero Gaussian measure if every continuous linear function f on B has a mean zero Gaussian distribution with variance $\int_B [f(x)]^2\,\mu(dx)$. The bilinear function T defined on B^* by

$$T(f, g) = \int_B f(x)g(x)\,\mu(dx) \qquad (f, g \in B^*)$$

is called the covariance function of μ. It is well known that a mean zero Gaussian measure on B is uniquely determined by its covariance function.

However, a mean zero Gaussian measure μ on B is also determined by a unique subspace H_μ of B which has a Hilbert space structure. The norm on H_μ will be denoted by $\|\cdot\|_\mu$ and it is known that the B norm $\|\cdot\|$ is weaker than $\|\cdot\|_\mu$ on H_μ. In fact, $\|\cdot\|$ is a measurable norm on H_μ in the sense of [3]. Since $\|\cdot\|$ is weaker than $\|\cdot\|_\mu$ it follows that B^* can be linearly embedded into the dual of H_μ, call it H_μ^*, and identifying H_μ with H_μ^* in the usual way we have $B^* \subseteq H_\mu \subseteq B$. Then by the basic result in [3] the measure μ is the extension of the canonical normal distribution on H_μ to B. We describe this relationship by saying *μ is generated by H_μ*. For details on these matters as well as additional references see [3] and [4].

2. **The basic inequality.** The norm $\|\cdot\|$ on B is *twice directionally differentiable* on $B-\{0\}$ if for $x, y \in B$, $x+ty \neq 0$, we have

$$(2.1) \qquad (d/dt)\,\|x + ty\| = D(x + ty)(y)$$

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where $D:B-\{0\}\to B^*$ is measurable from the Borel subsets of B generated by the norm topology to the Borel subsets of B^* generated by the weak-star topology, and

$$(2.2) \qquad (d^2/dt^2)\,\|x+ty\| = D^2_{x+ty}(y, y)$$

where D^2_x is a bounded bilinear form on $B\times B$. We call D^2_x the *second directional derivative* of the norm, and without loss of generality we can assume D^2_x is a symmetric bilinear form. That is, if T_x is a bilinear form which satisfies (2.2) then $\Lambda_x(y, z)=[T_x(y, z)+T_x(z, y)]/2$ also satisfies (2.2) and Λ_x is symmetric. Hence in all that follows we assume D^2_x is a symmetric bilinear form. Of course, if the norm is actually twice Fréchet differentiable on B with second derivative at x given by Λ_x, then it is well known that Λ_x is a symmetric bilinear form on $B\times B$, and in this case D^2_x would be equal to Λ_x since symmetric bilinear forms are uniquely determined on the diagonal of $B\times B$.

If $D^2_x(y, y)$ is continuous in x ($x\neq 0$) and for all $r>0$ and $x, h\in B$ such that $\|x\|\geqq r$ and $\|h\|\leqq r/2$ we have

$$(2.3) \qquad |D^2_{x+h}(h, h) - D^2_x(h, h)| \leqq C_r\,\|h\|^{2+\alpha}$$

for some fixed $\alpha>0$ and some constant C_r we say *the second directional derivative is* Lip(α) *away from zero.*

We now can state our main result.

THEOREM 2.1. *Let B denote a real separable Banach space with norm $\|\cdot\|$. Let $\|\cdot\|$ be twice directionally differentiable on B with the second derivative D^2_x being* Lip(α) *away from zero for some $\alpha>0$ and such that $\sup_{\|x\|=1}\|D^2_x\|<\infty$. Let $X_1, X_2, \cdots$ be independent B-valued random variables such that for some $\delta>0$*

$$(2.4) \qquad \sup_k E\,\|X_k\|^{2+\delta} < \infty, \qquad EX_k = 0 \qquad (k = 1, 2, \cdots)$$

and having common covariance function $T(f, g)=E(f(X_k)g(X_k))$ ($f, g\in B^*$). Then, if T is the covariance function of a mean zero Gaussian measure μ on B, it follows for $t\geqq 0$ and any $\beta>0$ that

$$(2.5) \qquad P\left(\left\|\frac{X_1 + \cdots + X_n}{\sqrt{n}}\right\| \geqq t\right) \leqq 2\mu(x\colon \|x\| \geqq t - \beta) + O(n^{-\min(\alpha,\delta)/2})$$

where the bounding constant is uniform in $t\geqq 2\beta$.

The proof of Theorem 2.1 uses a method which is due to Trotter [7]. The application of Trotter's method in this setting depends on a number of important relationships between H_μ and B as well as some of the nontrivial properties of Gaussian measures on B. The details of the proof are lengthy and will be presented in [**6**].

3. **Applications of the basic inequality.** Using the inequality of Theorem 2.1 we can obtain the central limit theorem and the law of the iterated logarithm for a sequence of B-valued random variables.

THEOREM 3.1. *Let B and $\{X_k\}$ satisfy the conditions in Theorem 2.1, and assume μ is a Gaussian measure on B with covariance function T. Then, if μ_n denotes the measure induced on B by $(X_1+\cdots+X_n)/\sqrt{n}$, we have $\lim_n \mu_n=\mu$ in the sense of weak convergence.*

The proof of Theorem 3.1 is not difficult and the main idea is to use (2.5) to prove that for each $\varepsilon>0$ there is a finite dimensional subspace E of B such that

$$\mu_n(E^\varepsilon) > 1 - \varepsilon \qquad (n \geqq 1). \tag{3.1}$$

Here E^ε is the ε neighborhood of E in B. Since the finite dimensional distributions of the sequence $\{\mu_n\}$ converge to those of μ, (3.1) is then sufficient for the conclusion of Theorem 3.1.

We now turn to the law of the iterated logarithm. LLn denotes $\log\log n$ if $n\geqq 3$ and 1 for $n=1, 2$.

THEOREM 3.2. *Let B and $\{X_k\}$ satisfy the conditions in Theorem 2.1, and assume μ is a Gaussian measure on B with covariance function T. If K is the unit ball of the Hilbert space H_μ which generates μ, then*

$$P\left(\lim_n \left\| \frac{X_1+\cdots+X_n}{(2n\, LLn)^{1/2}} - K \right\| = 0\right) = 1 \tag{3.2}$$

and

$$P\left(C\left(\left\{\frac{X_1+\cdots+X_n}{(2n\, LLn)^{1/2}}\right\}\right) = K\right) = 1 \tag{3.3}$$

where $C(\{a_n\})$ denotes the cluster set of the sequence $\{a_n\}$.

It is known that K is a compact subset of B; thus (3.2) implies that with probability one the sequence $\{(X_1+\cdots+X_n)/(2n\, LLn)^{1/2}\}$ is conditionally compact in B.

The proofs of (3.2) and (3.3) rest heavily on the inequality (2.5) and also on some of the nontrivial properties of Gaussian measures on B. The details will be given in [**6**].

Strassen's functional form of the law of the iterated logarithm for B-valued random variables can also be proved in this setting using (2.5) and the techniques developed in [**5**] where B was assumed to be a real separable Hilbert space.

4. Some spaces with smooth norm. Here we provide some examples of Banach spaces to which the above results apply. (S, Σ, m) denotes a measure space and m is a positive measure on (S, Σ).

THEOREM 4.1. *If $p \geqq 2$ and if for $x \in L^p(S, \Sigma, m)$ we define $\|x\| = \{\int_S |x(s)|^p m(ds)\}^{1/p}$, then the norm $\|\cdot\|$ has two directional derivatives and the second derivative is* Lip(α) *away from zero with $\alpha = 1$ for $p=2$ or $p \geqq 3$ and $\alpha = p-2$ for $2 < p < 3$. Furthermore,* $\sup_{\|x\|=1} \|D_x^2\| \leqq 2(p-1)$.

The results of Theorem 4.1 are suggested by those in [**1**], but do not seem to be immediate corollaries of [**1**]. Their proof, however, is rather straightforward. Furthermore, the derivatives in Theorem 4.1 are actually Fréchet derivatives.

Using Theorem 4.1 and assuming (S, Σ, m) is a σ-finite measure space we see that the L^p spaces $(2 \leqq p < \infty)$ satisfy the conditions used above. Thus the central limit theorem and the law of the iterated logarithm are valid in these spaces. A central limit theorem for random variables with values in an L^p space $(2 \leqq p < \infty)$ was previously known and appears in [**2**], but the log log law for non-Gaussian random variables is new for $p > 2$.

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