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NORMAL FIELD EXTENSIONS K/k AND K/k-BIALGEBRAS[1]

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Throughout the paper K/k is a field extension and p is the exponent characteristic.

In this paper I introduce the notion of K/k-bialgebra (coalgebra over K and algebra over k) and describe a theory of finite dimensional normal field extensions K/k based on a K-measuring K/k-bialgebra $H(K/k)$ (see 1.2, 1.6 and 1.10). This approach to studying K/k was inspired by my conviction that a successful theory would, in view of the Jacobson-Bourbaki correspondence theorem, result from suitably equipping the endomorphism ring $\mathrm{End}_k K$ of K/k with additional structure which would effectively reflect the multiplicative structure of K.

Some initial parts of the theory developed here are parallel to Moss Sweedler's very effective theory of normal extensions based on a universal cosplit K-measuring k-bialgebra (coalgebra over k and algebra over k) [**1**].

In §1 the structure of K/k is related to that of $H(K/k)$. At the same time, general properties of K/k-bialgebras are described. In §2, K-measuring k-bialgebras and semilinear K-measuring K/k-bialgebras are related, and the structure of semilinear conormal K-measuring K/k-bialgebras is described. In §3 the structure of a finite dimensional radical extension K/k and that of its K/k-bialgebra $H(K/k)$ are described in detail in terms of the toral k-subbialgebra T of $H(K/k)$. As an application of the theory of toral subbialgebras, a generalization of a theorem of Jacobson on finite dimensional Lie algebras of derivations of a field K is given in §4.

The material outlined in this paper is the outgrowth of preliminary research described at the 1971 Ohio State University Conference on Lie Algebras and Related Topics. A complete development of this material is given in a forthcoming book [**2**].

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1. **K/k-bialgebras and $H(K/k)$.** The ring $\mathrm{End}_k K$ of k-linear endomorphisms of a field extension K/k can be regarded as a K/k-algebra in the sense of the following definition.

1.1. DEFINITION. A *K/k-algebra* is a vector space A over K together with a mapping $\pi: A \otimes_k A \to A$ which is K-linear, $A \otimes_k A$ being regarded as vector space over K via the left hand factor, such that A together with π is a k-algebra (associative algebra with idenity over k).

1.2. DEFINITION. $H(K/k)$ is the union of all coclosed subsets of $\mathrm{End}_k K$, "coclosed" being defined as follows.

1.3. DEFINITION. A subset C of $\mathrm{End}_k K$ is *coclosed* if for each $x \in C$, there exist elements ${}_1x, x_1, \cdots, {}_nx, x_n \in C$ such that $x(ab) = \sum_i {}_ix(a)x_i(b)$ for all $a, b \in K$.

1.4. PROPOSITION. *$H(K/k)$ is a coclosed K-subspace of* $\mathrm{End}_k K$ *and a subalgebra of* $\mathrm{End}_k K$ *as k-algebra.*

By the above proposition, we may regard $H(K/k)$ as K/k-algebra.

1.5. THEOREM. *There exist K-linear mappings $\Delta: H(K/k) \to H(K/k) \otimes_K H(K/k)$ and $\varepsilon: H(K/k) \to K$ uniquely determined by the conditions:*

1. *for $x \in H(K/k)$ and ${}_1x, x_1, \cdots, {}_nx, x_n \in H(K/k)$, $\Delta(x) = \sum_i {}_ix \otimes x_i$ if and only if $x(ab) = \sum_i {}_ix(a)x_i(b)$ for all $a, b \in K$;*
2. *$\varepsilon(x) = x(1_k)$ for all $x \in H(K/k)$, 1_k being the identity of K.*

1.6. THEOREM. *$H(K/k)$ as K/k-algebra together with the mappings Δ, ε is a K/k-bialgebra in the sense of the following definition.*

1.7. DEFINITION. A *K/k-bialgebra* is a K/k-algebra H together with mappings $\Delta: H \to H \otimes_K H$ and $\varepsilon: H \to K$ such that H together with Δ and ε is a K-coalgebra and

1. $\Delta(1_H) = 1_H \otimes 1_H$;
2. $\Delta(xy) = \sum_{i,j} {}_ix_jy \otimes x_iy_j$ whenever $x, y \in H$, $\Delta(x) = \sum_i {}_ix \otimes x_i$ and $\Delta(y) = \sum_j {}_jy \otimes y_j$;
3. $\varepsilon(1_H) = 1_K$;
4. $\varepsilon(xy) = \varepsilon(x)\varepsilon(y)$ for all $x, y \in H$ such that $\varepsilon(y) \in k$.

A *k-bialgebra* is a k/k-bialgebra. A *subbialgebra* respectively *bi-ideal* of a K/k-bialgebra (or k-bialgebra) H is a subring and subcoalgebra D/ideal and coideal P of H.

Obviously, D and H/P are K/k-bialgebras (k-bialgebras).

1.8. THEOREM. *If the dimension $K{:}k$ of K over k is finite, then $H(K/k) =$* $\mathrm{End}_k K$.

The inclusion mapping $i: H(K/k) \to \mathrm{End}_k K$ is a measuring representation of $H(K/k)$ on K in the following sense.

1.9. DEFINITION. A *measuring representation* of a K-coalgebra H on a K/k-algebra A is a K-linear mapping $\rho: H \to \mathrm{End}_k A$ such that $\rho(x)(1_A) = \varepsilon(x)1A$ and $\rho(x)(ab) = \sum_i \rho({}_ix)(a)\rho(x_i)(b)$ for $x \in H$ and $a, b \in A$. A *measuring representation* of a K/k-bialgebra (k-bialgebra) H on a K/k-algebra (k-algebra) A is a mapping $\rho: H \to \mathrm{End}_k A$ which is a representation of H as k-algebra and a measuring representation of H as K-coalgebra (k-coalgebra).

$H(K/k)$ together with i is a K-measuring K/k-bialgebra in the following sense.

1.10. DEFINITION. A *K-measuring K/k-bialgebra* (*k-bialgebra*) is a K/k-bialgebra (K-bialgebra) H together with a measuring representation ρ of H on K. The shorthand notation $\rho(x)(a) = x(a)$ for $a \in K$, $x \in H$ is often used for measuring bialgebras (H, ρ).

For any K-measuring K/k-bialgebra (k-bialgebra) (H, ρ), let K^H be the subfield $\{a \in K \mid \rho(x)(ab) = a\rho(x)(b)$ for all $b \in K$ and all $x \in H\}$ and let Kern $H = \{x \in H \mid \rho(x) = 0\}$.

1.11. THEOREM. *Let H be a K-measuring K/k-bialgebra. Then* Kern H *is a bi-ideal of H. If $K:k < \infty$, then $H/$*Kern *H is isomorphic as K/k-bialgebra to $H(K/K^H)$.*

The above theorem has no natural counterpart for K-measuring k-bialgebras H, since Kern H is not always a bi-ideal of H.

Let $\mathscr{F} = \{k' \mid k'$ is a subfield of K containing k and $K:k' < \infty\}$ and $\mathscr{S} = \{H \mid H$ is a subbialgebra of $H(K/k)$ and $H:K < \infty\}$.

1.12. THEOREM. *$\mathscr{F}$ is mapped bijectively to $\mathscr{S}$ by the mapping $k' \mapsto H(K/k')$.*

1.13. THEOREM. *For $K:k < \infty$, K/k is normal respectively radical respectively Galois if and only if $H(K/k)$ is conormal respectively coradical respectively co-Galois in the sense of* 1.15 *below.*

1.14. DEFINITION. A K-coalgebra H is *colocal* respectively *cosemisimple* respectively *cosplit* respectively *cocommutative* if H has a unique minimal nonzero subcoalgebra respectively H is the sum of its minimal nonzero subcoalgebras respectively every minimal nonzero subcoalgebra of H is one dimensional respectively $\Delta(x) = \sum_i {}_ix \otimes x_i$ if and only if $\Delta(x) = \sum_i x_i \otimes {}_ix$ for all $x \in H$, that is, if the dual K-algebra H^* of H is local respectively semisimple respectively split respectively commutative. (Here, H^* is *semisimple* if every finite dimensional homomorphic image is a direct sum of fields.)

1.15. DEFINITION. A K/k-bialgebra H is *conormal* if H is cosplit and cocommutative and the semigroup $G(H)$ of grouplike elements of H is a group. If H is conormal, H is *co-Galois* respectively *coradical* if H is cosemisimple respectively colocal, that is, if

$$H = KG(H)\ (K\text{-span of } G(H))/G(H) = \{1_H\}.$$

1.16. THEOREM. *A K/k-bialgebra H has a unique maximal colocal subbialgebra $H(1_H)$.*

1.17. THEOREM. *Let K/k be finite dimensional and normal. Then $K = K_{\mathrm{Gal}}K_{\mathrm{rad}}$ (internal tensor product of k-algebras) and $H(K/k) = H_{\mathrm{Gal}}H_{\mathrm{rad}}$ (internal tensor product of k-algebras) where K_{Gal} and K_{rad} are Galois and radical extensions of k respectively, H_{Gal} and H_{rad} are K_{Gal}/k- and K_{rad}/k-subbialgebras of H respectively, in the sense of* 1.18 *below, H_{Gal} and H_{rad} stabilize K_{Gal} and K_{rad} respectively and the mappings $x \mapsto x|_{K_{\mathrm{Gal}}}$ and $y \mapsto y|_{K_{\mathrm{rad}}}$ map H_{Gal} and H_{rad} isomorphically to $H(K_{\mathrm{Gal}}/k)$ and $H(K_{\mathrm{rad}}/k)$ respectively.*

A subset C of a K/k-bialgebra H is *coclosed* if for each $x \in C$, there exist ${}_1x, x_1, \cdots, {}_nx, x_n \in C$ such that $\Delta(x) = \sum_i {}_ix \otimes_K x_i$. A k'-subspace C of H is *linearly disjoint* to K over k' if a k'-basis for C is a k-basis for the K-span KC of C, k' being a subfield of K containing k.

1.18. DEFINITION. A k'-subcoalgebra of a K/k-bialgebra H is a coclosed k'-subspace H' of H containing 1_H which is linearly disjoint to K over k' and satisfies the condition $\varepsilon(H') \subset k'$. A *$k'/k$-subbialgebra* respectively *k-subbialgebra* of H is a subring of H which is also a k'-subcoalgebra respectively k-subcoalgebra of H.

1.19. PROPOSITION. *A k'-subcoalgebra respectively k'/k-subbialgebra respectively k-subbialgebra of a K/k-bialgebra H can be regarded naturally as a k'-coalgebra respectively k'/k-bialgebra respectively k-bialgebra.*

1.20. THEOREM. *For any finite dimensional normal extension K/k and for $H = H(K/k)$, $H(1_H) = H(K/K_{\mathrm{Gal}})$ and $KG(H) = H(K/K_{\mathrm{rad}})$. Moreover, H_{rad} and H_{Gal} are K_{rad}- and K_{Gal}-forms of the K/k-bialgebras $H(1_H)$ and $KG(H)$ respectively, in the following sense.*

1.21. DEFINITION. A *k'-form/k-form* of a K/k-bialgebra H is a k'/k-subbialgebra respectively k-subbialgebra H' of H such that $H = KH'$ (K-span of H').

1.22. THEOREM. *Let K/k be finite dimensional and normal. Then the cosplit k-forms H of $H(K/k)$ which stabilize K_{rad} and K_{Gal} are those of the form $H = H_{\mathrm{rad}}(kG)$ (internal tensor product of k-bialgebras) where H_{rad} is a k-form of $H(K/K_{\mathrm{Gal}})$ and G is the group of automorphisms of K/k.*

In particular, the problem of finding a k-form for $H(K/k)$ for K/k finite *dimensional and normal reduces to the same problem for K/k finite dimensional and radical.*

2. **The structure of conormal K-measuring K/k-bialgebras.** Let H_k be a k-bialgebra and ρ a measuring representation of H_k on a k-algebra A. Then $A \otimes_k H_k$ can be regarded as k-algebra with product

$$(a \otimes x)(b \otimes y) = \sum_i a_i x(b) \otimes x_i y \qquad (a, b \in A,\ x, y \in H_k),$$

called the *semidirect product* (smash product) of A and H.

2.1. PROPOSITION. *Let (H_k, ρ_k) be a K-measuring k-bialgebra. Then $(K \otimes_k H_k, \mathrm{id}_K \otimes \rho_k)$ together with the semidirect product k-algebra structure and obvious K-coalgebra structure for $K \otimes_k H_k$ is a K-measuring K/k-bialgebra which is semilinear in the sense that $x(by) = \sum_i {}_i x(b) x_i y$ for all $b \in K$, $x, y \in K \otimes_k H_k$.*

2.2. PROPOSITION. *Let (H, ρ) be a semilinear K-measuring K/k-bialgebra. Let H_k be a k-form of H and let $\rho_k = \rho|_{H_k}$. Then (H_k, ρ_k) is a K-measuring k-bialgebra and (H, ρ) is isomorphic to $(K \otimes_k H_k, \mathrm{id}_K \otimes \rho_k)$.*

2.3. DEFINITION. Let C_K be a K-coalgebra, C_k a k-coalgebra. Then one can construct the *tensor product K-coalgebra $C_K \otimes_k C_k$*. If H_K is a K/k-bialgebra and H_k a k-bialgebra, the *tensor product K/k-bialgebra $H_K \otimes_k H_k$* has the tensor product k-algebra and K-coalgebra structures.

2.4. DEFINITION. Let H be a K/k-bialgebra. Let H_K be a K/k-subbialgebra of H and H_k a k-subbialgebra of H. Then we say that H is the *internal semidirect product* of H_K and H_k or that $H = H_K H_k$ (*internal semidirect product* K/k-bialgebra) if there exists a measuring representation ρ of H_k on H_K such that the K-linear mapping $H_K \otimes_k H_k \to H$ induced by the product in H is an isomorphism (of k-algebra and K-coalgebras) from $H_K \otimes_k H_k$ (semidirect product k-algebra with respect to ρ and tensor product K-coalgebra).

The following theorem generalizes to K/k-bialgebras a theorem due to Bertram Kostant [1] on k-bialgebras.

2.5. THEOREM. *Let H be a conormal semilinear K-measuring K/k-bialgebra. Then $H = H(1_H)kG(H)$ (internal semidirect product K/k-bialgebra) where $kG(H)$ is the k-span of $G(H)$.*

2.6. DEFINITION. A K-measuring K/k-bialgebra (H, ρ) is $G(H)$-*faithful* if the restriction of ρ to $G(H)$ is injective.

If K_{rad}/k and K_{Gal}/k are finite dimensional radical and Galois extensions respectively, H_{rad} is a coradical K_{rad}-measuring K_{rad}/k-bialgebra and H_{Gal} is a co-Galois $G(H_{\mathrm{Gal}})$-faithful K_{Gal}-measuring K_{Gal}/k-bialgebra, then

$H=H_{\mathrm{rad}} \otimes_k H_{\mathrm{Gal}}$ can be regarded naturally as conormal $G(H)$-faithful K-measuring K/k-bialgebra where $K=K_{\mathrm{rad}} \otimes_k K_{\mathrm{Gal}}$.

The following theorem generalizes 1.17.

2.7. THEOREM. *The finite dimensional conormal $G(H)$-faithful semilinear measuring bialgebras H are precisely the $H_{\mathrm{rad}} \otimes_k H_{\mathrm{Gal}}$ described above.*

3. The toral structure of a radical extension K/k and its K/k-bialgebra $H(K/k)$.

Let K/k be finite dimensional.

3.1. DEFINITION. A k-subcoalgebra (k-subbialgebra) T of $H(K/k)$ is *diagonalizable* respectively *toral* if $t^p \in T$ for all $t \in T$, $st=ts$ for all $s, t \in T$ and each element of T is diagonalizable respectively semisimple as linear transformation of K over k.

3.2. THEOREM. *There is a bijective correspondence between the diagonalizable k-subbialgebras of $H(K/k)$ and the decompositions $K=\sum_{i\in S} K_i$ (direct sum of k-subspaces) such that $\{K_i | i \in S\}$ is a group under the composition $K_iK_j=k$-span of $\{xy | x \in K_i, y \in K_j\}$.*

3.3. THEOREM. *$K=k(x_1)\cdots k(x_n)$ (internal tensor product of k-algebras where $x_i^e \in k$ $(1\leqq i\leqq n)$ for some integer $e>0$ if and only if $K^T=k$ for some diagonalizable k-subbialgebra of $H(K/k)$.*

Assume throughout the remainder of the section that K/k is radical. Let L be the separable closure of k, $\bar{K}=L\otimes_k K$, $\bar{k}=L\otimes_k k$, $\bar{T}=L\otimes_k T$ for any vector space T over k. Let the group G of automorphisms of L/k act on $\bar{K}$, $\bar{k}$, $\bar{T}$ by $g(a\otimes b)=g(a)\otimes b$ for $g \in G$. Identify $H(K/k)\otimes_k T$ and $H(\bar{K}/\bar{k})$.

3.4. THEOREM. *The set of toral k-subcoalgebras (k-subbialgebras) of $H(K/k)$ is mapped bijectively to the set of G-stable diagonalizable k-subcoalgebras (k-subbialgebras) of $H(\bar{K}/\bar{k})$ under $T\mapsto\bar{T}$, the inverse being $\bar{T}\mapsto\bar{T}^G$ (fixed points of G in $\bar{T}$).*

3.5. THEOREM. *Let T be a toral k-subbialgebra of $H(K/k)$. Then the centralizer $H(K/k)^T=\{x \in H(K/k) | xt=tx \text{ for all } t \in T\}$ of T in $H(K/k)$ is a K^T-form of $H(K/k)$.*

The above theorem implies that $H(K/k)^T$ is a K^T-measuring K^T/k-bialgebra with respect to the measuring representation $\rho: H(K/k)^T\to \mathrm{End}_k K^T$, ρ being restriction to K^T.

3.6. THEOREM. *For any toral k-subbialgebra T of $H(K/k)$, $H(K/K^T)=KT$ (K-span of T) and $H(K^T/k)\cong H(K/k)^T/I$ where I is the bi-ideal $\{x \in H(K/k)^T : x|_K T=0\}$.*

4. **Lie p-subcoalgebras of $H(K/k)$.** Let K/k be a (possibly infinite dimensional) field extension.

4.1. DEFINITION. A *Lie p-subcoalgebra* of $H(K/k)$ is a K-subcoalgebra C of $H(K/k)$ such that $[x, y]=xy-yx$ and x^p are elements of C for all $x, y \in C$.

4.2. THEOREM. *Let C be a finite dimensional colocal K-subcoalgebra of $H(K/k)$. Then $K^{p^n} \subset K^C$ for some n.*

4.3. THEOREM. *Let C be a finite dimensional Lie p-subcoalgebra of $H(K/k)$. Then $K{:}K^C < \infty$.*

The above theorem is proved by induction, using a more general version of Theorem 3.5.

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