APPROXIMATION NUMBERS AND KOLMOGOROFF DIAMETERS OF BOUNDED LINEAR OPERATORS

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0. Introduction. In this abstract we state some results concerning the approximation numbers and Kolmogoroff diameters of various operators between Banach spaces. Let $\mathscr{L}(E, F)$ and $\mathscr{F}(E, F)$ denote the bounded linear operators and the closure of the finite rank operators, respectively.

Following Pietsch [8], the nth approximation number $\alpha_n(T)$, of a $T \in \mathscr{L}(E, F)$ is defined as follows:

$$\alpha_n(T) = \inf\{\|T - A\| : \operatorname{rank} A \leqq n\};$$

the nth Kolmogoroff diameter of a $T \in \mathscr{L}(E, F)$ is defined by

$$d_n(T) = \inf\{\|Q_G T\| : \dim G \leqq n\}.$$

Here the infimum is over all subspaces $G \subseteq F$ and Q_G denotes the canonical quotient map of $F \rightarrow F/G$.

It is clear that $\alpha_n(T)$ and $d_n(T)$ are monotone decreasing sequences and that $\lim_n \alpha_n(T) = 0$ if and only if T is the limit of finite rank operators and $\lim_n d_n(T) = 0$ if and only if T is compact.

For a brief discussion of the algebraic and analytic properties of α_n and d_n (and other characteristics of bounded linear operators) see [7]. For compact operators on Hilbert space $\alpha_n(T) = d_n(T)$ and this characteristic has been extensively studied in the book of Gohberg and Kreĭn [1]. For arbitrary Banach spaces the results are few. Some very interesting results can be found in the papers [5] and [6]; related results are also to be found in the classic memoir of Grothendieck, second part [2].

Except where noted all our notation is standard. Details of the following results will appear elsewhere.

1. General relationships between α_n and d_n.

1.1. THEOREM. *For any $T \in \mathscr{L}(E, F)$ the following inequality is valid*:

$$d_n(T) \leqq \alpha_n(T) \leqq (\sqrt{n} + 1)d_n(T), \quad \text{for each } n.$$

The best value, $p(n)$, for which $\alpha_n(T) \leqq p(n)d_n(T)$ is not known. In general, $p(n)$ cannot be replaced by a constant K.

If the domain space is nice enough, however, α_n and d_n always coincide. More precisely,

1.2. THEOREM. *If E is an $L_1(\mu)$-space then for any $T \in \mathscr{L}(E, F)$, F arbitrary, $\alpha_n(T)=d_n(T)$ for each n. In particular, if E is a Banach space and q a norm one surjection of $L_1(\mu)$ onto E, then for any $T \in \mathscr{L}(E, F)$, $d_n(T)=\alpha_n(Tq)$ for each n.*

2. Diagonal operators on the l_p-spaces.
An operator T of the form $Te_n=\lambda_n e_n$ where (e_n) denotes the unit vectors in any l_p-space is called a diagonal operator and we say that T corresponds to (λ_n).

2.1. THEOREM. *Suppose (λ_n) is an increasing sequence with limit λ. Then*

(1) *if $T:l_1 \to l_\infty$ corresponds to (λ_n), $d_n(T)=\alpha_n(T)=\lambda/2$ for all n; and, if $T:l_1 \to c_0$ then $d_n(T)=\alpha_n(T)=\lambda$ for all n;*

(2) *if $T:l_p \to l_\infty$, $p>1$, $\alpha_n(T)=\lambda$ for all n and $d_n(T)=\lambda/2^{1/p}$ for all n;*

(3) *if $1 \leqq p \leqq q < \infty$ and $T:l_p \to l_q$ then $d_n(T)=\alpha_n(T)=\lambda$ for all n.*

If one considers the natural injection I of $l_1 \to c_0$, (1) above shows that $\alpha_n(I) \neq \alpha_n(I^*)$ and (2) above shows that, in general, $\alpha_n(T) \neq d_n(T)$.

2.2. THEOREM. *Suppose (λ_n) is a decreasing sequence.*

(4) *If $1 \leqq q < p < \infty$ and $T:l_p \to l_q$ corresponds to (λ_n) then*

$$d_n(T) = \alpha_n(T) = \left(\sum_{n+1}^{\infty} |\lambda_i|^r \right)^{\frac{1}{r}}$$

where $1/r+1/p=1/q$;

(5) *If $1 \leqq p < q < \infty$ then*

$$\left(\sum_{1}^{n+1} \lambda_i^{-r} \right)^{\frac{1}{r}} \leqq \alpha_n(T) \leqq \min\left(\frac{\lambda_n}{2}, \lambda_{n+1} \right), \quad \frac{1}{r} + \frac{1}{q} = \frac{1}{p}.$$

We mention that (4) above for $p \neq \infty$ has been computed by P. Johnson [3] and asymptotic estimates for $p=\infty$ have been given by Pietsch [8]. We are unable to compute $\alpha_n(T)$ exactly in case (5). However, there are operators $T, S:l_1 \to l_\infty$ corresponding to decreasing sequences (λ_n) and

(β_n) respectively such that $\alpha_n(T)=(\sum_1^{n+1}\lambda_i^{-1})^{-1}$ and $\alpha_n(S)=\beta_n/2$ for each n. Thus (5) gives the best asymptotic values of α_n possible.

For the remaining case, we remark that it is well known (see, e.g., [7]) that for $T: l_p \to l_p$ corresponding to a decreasing sequence (λ_n), $d_n(T)=\alpha_n(T)=\lambda_{n+1}$ for each n.

3. *H-operators and* ε_p*-finite bases.* An operator T acting on a Banach space E is an *H-operator* if its spectrum is real and its resolvent $\rho(T)$ satisfies

$$\|(T-\lambda I)^{-1}\| \leqq C|I_m\lambda|^{-1} \qquad (I_m\lambda \neq 0), \quad C \text{ independent of } \rho(T).$$

An operator on a Hilbert space is an *H-operator* with constant $C=1$ if and only if it is a selfadjoint operator. Markus̆ [5] has proved that for a compact *H-operator* T with eigenvalues (λ_n) (numbered in order of decreasing modulus and taking into account their multiplicity), the sequences $(d_n(T))$, $(\alpha_n(T))$ and (λ_n) are equivalent, i.e., there are constants A, B such that $d_n(T)\leqq\alpha_n(T)\leqq A|\lambda_{n+1}|\leqq Bd_n(T)$.

We generalize this result to certain types of diagonal maps. We say that a (Schauder) basis (x_n) for a Banach space E is ε_p-*finite* provided $\varepsilon_p(x_n)= \sup_{\|f\|=1}(\sum_{n=1}^{\infty}|f(x_n)|^p)^{1/p}<+\infty$.

3.1. THEOREM. *Let* T *be a diagonal basis mapping* $e_n \to \lambda_n u_n$ *where* (e_n, f_n) *and* (u_n, g_n) *are Schauder bases for* E *and* F *respectively and* (λ_n) *is a monotonically decreasing null sequence,* $\lambda_n \neq 0$. *Suppose that* $\varepsilon_1(g_n \otimes e_n)$ *and* $\varepsilon_1(f_n \otimes u_n)$ *are finite. Then*

$$[\varepsilon_1(g_n \otimes e_n)]^{-1}\lambda_{n+1} \leqq d_n(T) \leqq \alpha_n(T) \leqq [\varepsilon_1(f_n \otimes u_n)]\lambda_{n+1}.$$

In any infinite dimensional Banach space there are basic sequences satisfying the hypotheses of 3.1.

Motivated by this result and a classical result of S. Bernstein, we say that two Banach spaces E and F form a *Bernstein pair* if for any positive monotonic null sequence (b_n) there is a $T \in \mathscr{L}(E, F)$ such that $(\alpha_n(T))$ is equivalent to (b_n).

3.2. THEOREM. *Let* E *and* F *be arbitrary infinite dimensional Banach spaces. Then there are infinite dimensional spaces* E_0 *and* F_0 *in* E *and* F *respectively such that* $\langle E_0, F_0 \rangle$ *is a Bernstein pair. Moreover, the operator* T *satisfying the definition for* $\langle E_0, E_0 \rangle$ *can be chosen to be a compact H-operator.*

3.3. PROPOSITION. *If* $E=E_1\oplus E_2$ *and* $F=F_1\oplus F_2$ *and* $\langle E_i, F_i \rangle$ *is a Bernstein pair for* $i=1$ *or* 2, *then* $\langle E, F \rangle$ *is a Bernstein pair.*

As we have observed, in general, $\alpha_n(T) \neq \alpha_n(T^*)$. Thus the next result is somewhat surprising.

3.4. THEOREM. *Suppose $T \in \mathscr{F}(E, F)$. Then $\alpha_n(T) = \alpha_n(T^*)$ for each n.*

Recall that an operator is of *type l_p* [8] if $(\alpha_n(T)) \in l_p$.

3.5. COROLLARY. *An operator T is of type l_p if and only if T^* is of type l_p.*

Another immediate corollary to 3.4 is the following fact concerning Bernstein pairs.

3.6. COROLLARY. *If $\langle E, F \rangle$ is a Bernstein pair, so is $\langle F', E' \rangle$.*

Also from 3.4 we obtain for operators in $\mathscr{F}(E, F)$ the dual result of 1.2.

3.7. COROLLARY. *Let i be an embedding of a Banach space F into $l_\infty(\Gamma)$. Then if $T \in \mathscr{F}(E, F)$ we have $\alpha_n(iT) = d_n(T)$ for all n.*

Perhaps all couplings of Banach spaces form Bernstein pairs. Indeed, all the classical Banach spaces form Bernstein pairs.

More precisely, let E and F be $\mathscr{L}_p$- and $\mathscr{L}_q$-spaces in the sense of Lindenstrauss and Pełczyński [4]. Then $\langle E, F \rangle$ is a Bernstein pair.

Finally, we introduce the notion of a weak Bernstein pair. Two Banach spaces E and F form a *weak Bernstein pair* if there is a decreasing sequence $(b_n) \in c_0 \backslash l_2$ and $T \in \mathscr{L}(E, F)$ such that $(\alpha_n(T))$ is equivalent to (b_n). Clearly a Bernstein pair is a weak Bernstein pair.

3.8. THEOREM. *Any of the following is sufficient in order that Banach spaces E and F form a weak Bernstein pair.*
  (1) *E contains an unconditional basic sequence;*
  (2) *E and F have the approximation property;*
  (3) *E and F admit equivalent uniformly convex norms.*

The importance of the notion of a weak Bernstein pair is shown by the following corollary which partially answers an unsolved problem of Grothendieck and Rosenthal.

3.9. COROLLARY. *If (E, E) forms a weak Bernstein pair then on E there is a compact nonnuclear operator.*

ADDED IN REMARKS. Since submitting this abstract we have received a preprint from Professor A. Pietsch entitled *s-numbers of bounded linear operators*. In this paper, he develops an axiomatic theory of *s*-numbers: the α_n and d_n above are special cases of this concept. Professor Pietsch proves many interesting results disjoint from our paper and also observes our Theorems 1.1 and 1.2 and the result of P. Johnson, 2.2(4).

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