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ON THE MAXIMUM *N*TH DIAMETER

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ABSTRACT. Herein is disproved, in some cases, a plausible conjecture on the maximum value of the Nth diameter of a closed bounded connected planar set of logarithmic capacity one.

Introduction. Let N be a positive integer greater than one. Define the Nth diameter of a closed bounded connected planar set to be the largest geometric mean of the distances between any N points of the set. This notion is a generalization of the usual diameter—which is just the case $N=2$. Then, if the above set is of logarithmic capacity one, how big can the Nth diameter get?

Fix the set in question. Let d_N denote the Nth diameter of the set. Then, as shown by M. Schiffer (in [4]),

$$d_N \leqq 4^{1/N} N^{1/(N-1)} \quad \text{for} \quad N = 2, 3.$$

This is the bound achieved only for the set consisting of the N-pronged slit forking at the origin and terminating at the Nth roots of four, or translations of the set. This set is a contender for the set with maximum Nth diameter, since the function of the form $az \perp$power series in $(1/z)$ which maps the exterior of the unit disc onto the exterior of the set is $(z^N \perp 2 \perp z^{-N})^{1/N}$, whose coefficient of the z term is exactly one.

Thus the following conjecture seems plausible:

For all N, $d_N \leqq 4^{1/N} N^{1/(N-1)}$. Equality is achieved iff the set, Γ, consists

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of N linear segments from the origin to the Nth roots of four, or translations of the set.

This diameter conjecture is, as mentioned above, true for $N=2$ and for $N=3$. It was, however, stated by P. R. Garabedian and M. M. Schiffer (in [**1**]) that the conjecture is false for $N=4$. Here, it will be shown that the conjecture is in fact false for all N such that N is an even integer greater than two. The other cases, when N is an odd integer greater than three, remain open.

Let Σ' denote the class of functions

$$G(z) = z + b_1 z^{-1} + b_2 z^{-2} + \cdots + b_n z^{-n} + \cdots$$

which are analytic and univalent for all z such that $|z|>1$. Let $Q(z)=$ that inverse function of $z^{-1}(1+z^N)^{2/N}$ which is in Σ'. Let $\lambda=$a positive real number greater than one. Then (G. Pólya and G. Szegö in [3]) a little reflection shows that, for any function $G(z)$ in Σ', the function

$$H(z) = \frac{4^{1/N}\lambda}{(\lambda^N + 1)^{2/N}} G\left[Q\left(\frac{(\lambda^N + 1)^{2/N}}{4^{1/N}\lambda} \frac{(1 + z^N)^{2/N}}{z}\right)\right]$$

is in Σ' and omits the N points

$$\frac{4^{1/N}\lambda}{(\lambda^N + 1)^{2/N}} G(\lambda\omega^k) \quad \text{with } \omega^N = 1, \text{ and } k = 1, 2, 3, \cdots, N.$$

Hence, if D_N denotes the largest possible value of the Nth diameter for any closed bounded connected planar set of logarithmic capacity one, then

$$D_N \geqq \frac{4^{1/N}\lambda}{(1 + \lambda^N)^{2/N}} \prod_{i \neq j} |G(\lambda\omega^i) - G(\lambda\omega^j)|^{1/(N^2-N)} \quad \text{where } \omega^N = 1.$$

Hence if the conjecture be true then

$$(1) \qquad \prod_{j \neq k} \left| \frac{G(\lambda\omega^j) - G(\lambda\omega^k)}{\lambda\omega^j - \lambda\omega^k} \right| \leqq (1 + 2\lambda^{-N} + \lambda^{-2N})^{N-1}.$$

Now define the double sequence of complex numbers $\{C_{P,Q}\}$ to be the Grunsky coefficients of $G(z)$ if

$$-\mathrm{Log}\left\{\frac{G(x) - G(y)}{x - y}\right\} = \sum_{P,Q \geqq 1} C_{P,Q} x^{-P} y^{-Q}.$$

Then on comparing both sides of (1), supposedly,

$$|C_{1,N-1} + C_{2,N-2} + C_{3,N-3} + \cdots + C_{N-1,1}| \leqq 2 - 2/N.$$

The following lemma thus completes the disproof of the conjecture, for N an even integer greater than two.

LEMMA. $|C_{1,2M-1} \perp C_{2,2M-2} \perp \cdots \perp C_{2M-1,1}| > 2-1/M$ *for some function* $G_M(z)$ *and for* $M=2, 3, 4, \cdots$.

PROOF. Let $M>1$ be fixed. Let

$$F(z) = z \perp a_2 z^2 \perp \cdots \perp a_n z^n \perp \cdots$$

be univalent for z such that $|z|<1$, and let

$$\begin{aligned} J(z) &= F^{-1/M}(z^{-M}) \\ &= z - M^{-1}a_2 z^{1-M} \perp (\tfrac{1}{2}M^{-1}a_2^2 \perp \tfrac{1}{2}M^{-2}a_2^2 - M^{-1}a_3)z^{1-2M} \perp \cdots . \end{aligned}$$

Then some Grunsky coefficients of $J(z)$ are

$$C_{k,2M-k} = C_{2M-k,k} = \tfrac{1}{2}M^{-1}a_2^2 \perp \tfrac{1}{2}kM^{-2}a_2^2 - M^{-1}a_3$$

where $k = 1, 2, 3, \cdots, M$

Therefore for $J(z)$

$$(2) \quad |C_{1,2M-1} \perp C_{2,2M-2} \perp \cdots \perp C_{2M-1,1}| = (2 - 1/M)\left|\left(a_3 - \frac{3M-1}{4M-2}a_2^2\right)\right|.$$

Now select $F(z)$ to maximize the quantity

$$(3) \qquad |a_3 - \alpha a_2^2| \quad \text{where } \alpha = \frac{3M-1}{4M-2}$$

over the set of functions with normalized power series which are univalent in the interior of the origin-centered unit disc. Then, as proven by G. M. Golusin (in [**2**]),

$$|a_3 - \alpha a_2^2| = 1 \perp 2\exp(-2\alpha/(1-\alpha)) \quad \text{for } \alpha \text{ in } (0, 1).$$

This, in conjunction with (2) and (3), proves the Lemma and the result.

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