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PSEUDO-INVERSES OF OPERATORS

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1. Let X and Y be complex Banach spaces, A a bounded linear operator from X to Y. If the null space $N(A)$ and the closed range $R(A)^-$ possess closed complementary subspaces U in X and V in Y respectively, the *pseudo-inverse* $A^\dagger$ *of* A *relative to* (U, V) is defined as the linear extension of $(A|U)^{-1}$ to $D(A^\dagger)=R(A)+V$ with the null space $N(A^\dagger)=V$. (This is a generalization to Banach space of the standard pseudo-inverse of a Hilbert space operator (cf. [**8**]). If $R(A)$ is closed, the definition agrees with the ones given in [**1**] and [7]. In this case $A^\dagger$ is defined and bounded on all of Y.) If $U=R(B)^-$ and $V=N(B)$ for some bounded linear operator $B: Y\to X$, $A^\dagger$ will be called the pseudo-inverse of A relative to B, written $A^{\dagger B}$. Proposition 6 of [**6**] leads to the following result.

THEOREM 1. *Suppose* $A:X\to Y$ *and* $B: Y\to X$ *are bounded linear operators such that* (a) $Y=R(A)^-\oplus N(B)$, (b) *the operator* $T=I-BA$ *is strongly power convergent* ($\{T^n\}$ *converges strongly*). *Then* $A^{\dagger B}$ *exists and is represented by*

$$A^{\dagger B}y = \sum_{n=0}^{\infty} (I - BA)^n By, \tag{1}$$

where the series converges in norm iff $y \in R(A)+N(B)$.

When T in Theorem 1 is uniformly power convergent ($\{T^n\}$ converges uniformly), then $R(A)$ is closed, (1) converges uniformly, and $A^{\dagger B}$ is defined and bounded on all of Y. In the case that A is an operator between Hilbert spaces, and $B=\alpha A^*$ with $0<\alpha<2\|A\|^{-2}$, Theorem 1 gives the well-known representation of the standard Hilbert space pseudo-inverse [**2**], [7], [**8**].

2. Let $A:X\to Y$ be a bounded linear operator between Banach spaces. A bounded linear operator $B: Y\to X$ is called a *pseudo-adjoint* of A if

$$X = N(A) \oplus R(B)^-, \qquad Y = R(A)^- \oplus N(B), \tag{2}$$

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and if there exists a real function h on $R(B)$ such that the operator $T=I-\alpha BA$ (with a suitable α) satisfies

$$(3)\qquad 0 < h(x) \leqq (\|x\|^2 - \|Tx\|^2)\|x\|^{-4} \qquad (x \neq 0), \qquad h(0) = 0,$$

$$(4)\qquad h(Tx) \geqq h(x).$$

The adjoint A^* of an operator A between Hilbert spaces is its pseudo-adjoint ($h(x)=\alpha(2-\alpha\|A\|^2)\|(A^*)^\dagger x\|^{-2}, 0<\alpha<2\|A\|^{-2}$). An idempotent operator A on a Hilbert space is its own pseudo-adjoint ($h(x)=\alpha(2-\alpha)\|x\|^{-2}, 0<\alpha<2$).

THEOREM 2. *Let B be a pseudo-adjoint of A (with $\alpha=1$ for simplicity). Then $T=I-BA$ is a strongly power convergent operator. For each $x \in R(B)^-$, $\|T^n x\|\to 0$ monotonically, and*

$$\|T^n x\|^2 \leqq \|x\|^2 (1 + nh(x)\|x\|^2)^{-1} \quad \textit{if } x \in R(B).$$

Proof is based on the inequality $\|T^{n+1}x\|^2 \leqq \|T^n x\|^2 - h(x)\|T^n x\|^4$ derived from (3) and (4) and the formula (4.11) of **[8]**. The next theorem generalizes Theorem 2(a) and (b) of **[8]** to operators between Banach spaces.

THEOREM 3. *Let B be a pseudo-adjoint of A (with $\alpha=1$). Then*

$$(5)\qquad \left\|\sum_{k=0}^{n}(I-BA)^k By - A^{\dagger B}y\right\|^2 \leqq \|A^{\dagger B}y\|^2(1 + nh(A^{\dagger B}y)\|A^{\dagger B}y\|^2)^{-1}$$

whenever the $R(A)^-$ component of y in $Y=R(A)^-\oplus N(B)$ lies in $R(AB)$. Moreover, the left-hand side of (5) converges monotonically to 0 for each $y \in R(A)+N(B)$.

3. Let $A:X\to Y$ be a bounded linear operator, and let U be a complement of $N(A)$ in X. The operator $A^\partial=(A|U)^{-1}$ will be called the *partial inverse of A relative to U.*

THEOREM 4. *Let $A:X\to Y$ and $B:Y\to X$ be bounded linear operators, with B bijective and such that $T=I-BA$ is strongly power convergent. Then A has the partial inverse A^∂ relative to $U=R(BA)^-$, represented by*

$$(6)\qquad A^\partial y = \sum_{n=0}^{\infty}(I-BA)^n By,$$

where the series converges iff $y \in R(A)$.

When the convergence of $\{T^n\}$ in the preceding theorem is uniform, $R(A)$ is closed, A^∂ bounded, and the series (6) converges uniformly on bounded sets of $R(A)$.

Both Theorems 1 and 4 can be applied to the approximate solution of the linear equation $Ax=y$ by means of the Picard iterations

$$x_{n+1} = (I - BA)x_n + By \qquad (x_0 \text{ given}). \tag{7}$$

In either case, if $y \in R(A)$, $\{x_n\}$ converges in norm to the solution $x=Px_0+A^{\partial}y$ of $Ax=y$, where Px_0 is the $N(A)$ component of x_0 in $X=N(A)\oplus R(BA)^{-}$. (In the case of Theorem 1, $A^{\partial}y=A^{\dagger B}y$ and $R(BA)^{-}=R(B)^{-}$.)

4. The strong power convergence of the operator $T:X\to X$ is the main hypothesis of Theorems 1 and 4. Various conditions for power convergence have been given in [2], [3], [4], [5]. It was shown in [5] that T is uniformly power convergent iff $\sigma(T)-\{1\}$ lies in the open unit disc and 1 is a pole of $(\lambda I-T)^{-1}$ of order $\leqq 1$ ($\sigma(T)$ denotes the spectrum of T). The following three results can be obtained from this theorem.

THEOREM 5. *Suppose $R(I-T)$ is closed and the continuous spectrum of T does not meet the unit circle. Then the weak, strong and uniform power convergence of T are all equivalent.*

The proof is based on the decomposition $T=T_0\oplus T_1$ of a weakly power convergent T, where $T_0=I|N(I-T)$ and $T_1=T|R(I-T)^{-}$ [6].

THEOREM 6. *Let T be power bounded, $R(I-T)$ closed, and let $I-T$ have finite descent. Then T is uniformly power convergent iff $\sigma(T)-\{1\}$ does not meet the unit circle.*

To prove Theorem 6, we show that $N((I-T)^2)=N(I-T)$ under the assumptions of the theorem.

The following result is a consequence of Theorems 5 and 6.

COROLLARY 1. *Suppose that T is power bounded and $f(T)$ compact, where f is a complex function analytic in an open neighborhood of $\sigma(T)$ with no zeros on $\sigma(T)-\{0\}$ such that* (a) $|f(\lambda)|<1$ *if* $|\lambda|<1$, (b) $f(1)=1$, *and* (c) $f'(1)\neq 0$. *Then T is weakly (=strongly=uniformly) power convergent iff $\sigma(T)-\{1\}$ does not meet the unit circle.*

The next three theorems give sufficient conditions of the Stein type (cf. [5]) for power convergence of Hilbert space operators. In the sequel, A, T and W are bounded linear operators on a Hilbert space H.

THEOREM 7. *Let $A=A^*$, and $A-T^*AT$ be positive definite on $R(I-T)^{-}$. Then the following conditions are equivalent:* (i) *$\{T^n\}$ converges uniformly,* (ii) *$\{T^n\}$ converges strongly,* (iii) *A is positive definite on $R(I-T)^{-}$.*

THEOREM 8. *Suppose the identity*

$$A - T^*AT = (I - T^*)W(I - T) \tag{8}$$

holds with A and W positive definite on H. Then T is strongly power convergent.

THEOREM 9. *Suppose the identity* (8) *holds with A and W positive definite on $R(I-T)^-$. If $I-T$ is an operator of finite descent, then T is uniformly power convergent.*

We outline the proof of the last theorem. We establish $N(I-T)\cap R(I-T)^-=\{0\}$ by showing that $(Ax, x)=(Ax, h)$ for each $x=(I-T)u+h$. Hence $X=N(I-T)\oplus R(I-T)$ with $R(I-T)$ closed. The rest is easy.

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