BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 80, Number 2, March 1974

FOLIATIONS AND GROUPS OF DIFFEOMORPHISMS

BY WILLIAM THURSTON

Communicated by William Browder, July 7, 1973

John Mather has described a close relation between framed codimension-one Haefliger structures (these form a class of singular foliations), and the group of compactly supported diffeomorphisms of $\boldsymbol{R}^1$, with discrete topology [**11**], [**12**], [**14**]. In this announcement I will describe generalizations of his ideas to higher codimension Haefliger structures and groups of diffeomorphisms of arbitrary manifolds. See Haefliger [**7**] for a development of Haefliger structures and their classifying spaces.

I would like to thank Boyd Anderson, André Haefliger and John Mather for long discussions and proddings about this material, and many others for helpful conversations and relevant information.

Let $\mathrm{Diff}^r(M^p)$ denote the group of C^r diffeomorphisms of M^p, a closed manifold. Let $\mathrm{Diff}_0^r(M^p)$ denote the connected component of the identity.

THEOREM 1. $\mathrm{Diff}_0^\infty(M^p)$ *is a simple group.*

The proof makes use of both the theorem of Epstein [**4**] that the commutator subgroup of $\mathrm{Diff}_0(M^p)$ is simple, and of the result of M. Herman [**9**] which gives the case M^p is a p-torus.

THEOREM 2. $B\bar{\Gamma}_p^\infty$ *is* $(p+1)$*-connected, where* $B\bar{\Gamma}_p^\infty$ *is the classifying space for framed, codimension* p, C^∞, *Haefliger structures.*

The more usual notation is $F\Gamma_p^\infty = B\bar{\Gamma}_p^\infty$. Haefliger proved [**6**] that $B\bar{\Gamma}_p^r$ is p-connected for $1 \leqq r \leqq \infty$; Mather proved that $B\bar{\Gamma}_1^\infty$ is 2-connected. Theorem 2 means that two C^∞ foliations of a manifold coming from nonsingular vector fields are homotopic as Haefliger structures if and only if the normal bundles are isomorphic.

Theorems 1 and 2 are proven by showing they are related; cf. Theorem 4 for a statement of a relationship.

COROLLARY. $P_1^{[p/2]}$ *is nontrivial in* $H^*(B\Gamma_p^\infty; \boldsymbol{R})$ *where* P_1 *is the first real Pontrjagin class of the normal bundle to the canonical Haefliger structure.*

AMS (MOS) subject classifications (1970). Primary 57D30, 57D50.

Thus, Bott's vanishing theorem [1], which says real Pontrjagin classes in $B\Gamma_p^r$ $(r\geqq 2)$ vanish above dimension $2p$, gives a sharp bound on dimensions.

This corollary in the case $p=2$ follows easily from Theorem 2.

For higher codimensions, product foliations then yield examples.

THEOREM 3. *$B\bar{\Gamma}_p^0$ is contractible.*

Again, Mather proved this when the codimension is one.

This means topological Haefliger structures are completely determined up to homotopy by their normal micro-bundles.

Theorem 3 implies that Bott's vanishing theorem is quite false in the topological case—any normal micro-bundle is the normal micro-bundle for a topological foliation. In fact, if the micro-bundle is differentiable, it even admits a Haefliger structure Lipschitz close to being differentiable.

A little background and notation is necessary before the statement of the more general relationships. Let G be a topological group. Let G_δ be G with discrete topology. Then the map $G_\delta\to G$ is a continuous map which has a homotopy-theoretic fiber $\bar{G}$. $\bar{G}$ is also a topological group: the explicit construction for $\bar{G}$ is the space of paths α in G ending at the identity $e=\alpha(1)$, with discrete topology on $\alpha(0)$. Then multiplication is pointwise. There are maps, now,

$$\bar{G}\to G_\delta\to G\to B\bar{G}\to BG_\delta\to BG,$$

and any two consecutive arrows define a fibration.

BG is the classifying space for G-bundles. BG_δ classifies flat G-bundles: for instance, $B\,\mathrm{Diff}^\infty(M^n)_\delta$ has an associated M-bundle, with discrete structure group: i.e., a C^∞ foliation transverse to the fibers of the bundle. Thus, $B\,\mathrm{Diff}^\infty(M^p)_\delta$ classifies "foliated M^p-bundles". Finally, $B\bar{G}$ classifies G-bundles with a flat structure, together with a global trivialization defined (up to homotopy); e.g. $B\,\overline{\mathrm{Diff}}^\infty(M^p)$ classifies "foliated M^p-products".

Let $\mathrm{Diff}_K(\boldsymbol{R}^p)$ be the group of diffeomorphisms of $\boldsymbol{R}^p$ with compact support. Then again, $B\,\overline{\mathrm{Diff}}_K^r(\boldsymbol{R}^p)\times\boldsymbol{R}^p$ has a foliation of codimension p transverse to the $\boldsymbol{R}^p$-factors. Thus, there is a classifying map

$$B\,\overline{\mathrm{Diff}}_K^r(\boldsymbol{R}^p)\times\boldsymbol{R}^p\to B\bar{\Gamma}_p^r.$$

(The image is in $B\bar{\Gamma}_p^r$ since there is a natural trivialization of the normal bundle to the foliation.)

The foliation agrees with the trivial, product foliation in a neighborhood of ∞ in the $\boldsymbol{R}^p$ factors. Thus, one obtains a map of the p-fold suspension of $B\,\overline{\mathrm{Diff}}_K^r\boldsymbol{R}^p$,

$$S^p(B\,\overline{\mathrm{Diff}}_K^r\boldsymbol{R}^p)\to B\bar{\Gamma}_p^r.$$

This defines an adjoint map $B\,\overline{\mathrm{Diff}}{}^r_K(\boldsymbol{R}^p)\to\Omega^p(B\bar{\Gamma}^r_p)$ to the p-fold loop space of $B\bar{\Gamma}^r_p$.

THEOREM 4. *The map* $B\,\overline{\mathrm{Diff}}{}^r_K(\boldsymbol{R}^p)\to\Omega^p(B\bar{\Gamma}^r_p)$ *induces an isomorphism on homology.*

This theorem is due to Mather in the case $p=1$.

The map is certainly not a homotopy equivalence since $\pi_1(B\,\overline{\mathrm{Diff}}{}^r_K\boldsymbol{R}^p)$ is highly nonabelian while $\pi_1(\Omega^p B\bar{\Gamma}^r_p)=\pi_{p+1}(B\bar{\Gamma}^r_p)$ is abelian.

Similarly, there is a map $B\,\overline{\mathrm{Diff}}{}^r(M^p)\times M^p\to B\Gamma^r_p$ which is a lifting of the classifying map for the tangent bundle of M^p, so there is a commutative diagram

$$\begin{tikzcd} & B\Gamma^r_p \arrow[d] \\ B\,\overline{\mathrm{Diff}}{}^r(M^p)\times M^p \arrow[ur] \arrow[r] & BO_p \end{tikzcd}$$

Let X be the space of liftings of the classifying map for $T(M^p)$ in BO_p to $B\Gamma^r_p$. Then we have a map $B\,\overline{\mathrm{Diff}}{}^r(M^p)\to X$.

THEOREM 5. *The map*

$$B\,\overline{\mathrm{Diff}}{}^r(M^p)\to X$$

induces an isomorphism on homology.

Again, this is not a homotopy equivalence since $\pi_1(X)$ is abelian.

For the case $r=0$, we assume M^p is a differentiable manifold.

COROLLARY. (a) $B\,\overline{\mathrm{Homeo}}(M^p)$ *is acyclic, where* $\mathrm{Homeo}(M^p)=\mathrm{Diff}^0(M^p)$ *is the group of homeomorphisms of* M^p.

(b) *The map* $B\,\mathrm{Homeo}(M^p)_\delta\to B\,\mathrm{Homeo}(M^p)$ *induces an isomorphism on homology.*

This corollary is implied by Theorems 3 and 5. Cf. Mather **[13]**, who showed $B\,\mathrm{Homeo}_K(\boldsymbol{R}^p)_\delta$ is acyclic.

COROLLARY. *The following groups are isomorphic, where k is the first positive integer such that one of them is nontrivial*:

(i) $H_k(B\,\overline{\mathrm{Diff}}{}^r(M^p);Z)$,

(ii) $H_k(B\,\overline{\mathrm{Diff}}{}^r_K(\boldsymbol{R}^p);Z)$,

(iii) $H_{k+p}(B\bar{\Gamma}^r_p;Z)$.

CONJECTURE. *This first k is* $p+1$, *for* $r=\infty$.

Mather's theorem **[11]** shows this for $p=1$. Bott and Haefliger showed that all differentiable characteristic classes (in some sense) vanish below this dimension, $H_{2p+1}(B\bar{\Gamma}_p^r; Z)$ [2], [3].

In [16] I sketched examples showing there is a surjective homomorphism

$$H_3(B\bar{\Gamma}_1^\infty; Z) \twoheadrightarrow \boldsymbol{R},$$

using the Godbillon-Vey invariant gv [5]. Recently I have extended this to arbitrary codimension, so there is a surjective homomorphism

$$H_{2p+1}(B\bar{\Gamma}_p^\infty; Z) \twoheadrightarrow \boldsymbol{R}.$$

Bibliography

1. R. Bott, *On a topological obstruction to integrability*, Proc. Int. Congress Nice, 1970, 27–36.

2. R. Bott and A. Haefliger, *On characteristic classes of Γ-foliations*, Bull. Amer. Math. Soc. **78** (1972), 1039–1044.

3. ———, *Continuous cohomology and characteristic classes*, (to appear).

4. D. B. A. Epstein, *The simplicity of certain groups of homeomorphisms*, Compositio Math. **22** (1970), 165–173. MR **42** #2491.

5. C. Godbillon and J. Vey, *Un invariant des feuilletage de codimension* 1, C.R. Acad. Sci. Paris Sér. A-B **273** (1971), A92–A95. MR **44** #1046.

6. A. Haefliger, *Feuilletages sur les variétés ouvertes*, Topology **9** (1970), 183–194. MR **41** #7709.

7. ———, *Homotopy and integrability*, Manifolds-Amsterdam 1970 (Proc. Nuffic Summer School), Lecture Notes in Math., vol. 197, Springer, Berlin, 1971, pp. 133–163. MR **44** #2251.

8. ———, *Sur les classes characteristiques des feuilletages*, Séminaire Bourbaki, No. 412, June 1972.

9. M. Herman, *Simplicité du groupe des difféomorphismes de classe* C^∞, *isotopes à l'identité, du tore de dimension n*, C.R. Acad. Sci. Paris Sér. A-B **273** (1971), A232–A234. MR **44** #4788.

10. M. Herman and F. Sergeraert, *Sur un théorème d'Arnold et Kolmogorov*, C.R. Acad. Sci. Paris Sér. A-B **273** (1971), A409–A411. MR **44** #7586.

11. J. Mather, *On Haefliger's classifying space*. I, Bull. Amer. Math. Soc. **77** (1971), 1111–1115. MR **44** #1047.

12. ———, *On Haefliger's classifying space*. II: *Approximation theorems*, (preprint).

13. ———, *The vanishing of homology of certain groups of homeomorphisms*, Topology **10** (1971), 297–298. MR **44** #5973.

14. ———, *Integrability in codimension* 1, Comment. Math. Helv. (to appear).

15. W. Thurston, *Noncobordant foliations of* S^3, Bull. Amer. Math. Soc. **78** (1972), 511–514. MR **45** #7741.

16. ———, *Variation of the Godbillon-Vey invariant in higher codimension*, (to appear).

Department of Mathematics, University of California, Berkeley, California 94720

School of Mathematics, Institute for Advanced Study, Princeton, New Jersey 08540

Current address: Department of Mathematics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139