BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 80, Number 2, March 1974

FINITE SUBGROUPS OF FINITE DIMENSIONAL DIVISION ALGEBRAS

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Communicated by George Seligman, September 29, 1973

Let D be a finite dimensional division algebra with center K and let G be a finite odd order subgroup of the multiplicative group D^* of D. This note is concerned with the following:

CONJECTURE. If K contains no nonidentity odd order roots of unity, then G is cyclic.

We announce here some results and raise several questions about this conjecture. In [3] and [4] we proved this conjecture if K is either an algebraic number field or the completion of an algebraic number field. (The converse is also proved in [4]; if K is an algebraic number field which does contain an odd order nonidentity root of unity, then there is a finite dimensional division algebra central over K containing a noncyclic odd order subgroup.) In this note we will consider the more general case where K is an arbitrary field of characteristic zero.

By a K-division ring we mean a finite dimensional division algebra with center K. Let G be a finite subgroup of the multiplicative group of a K-division ring D and, for L a subfield of D, denote by $\mathscr{V}_L(G)$ the division subring of D generated by L and G. Let $\mathscr{Z}_L$ denote the center of $\mathscr{V}_L(G)$ and e_L the exponent of $\mathscr{V}_L(G)$. The following result is basic to our approach to the above conjecture:

THEOREM 1. *With notation as above, let ζ be a primitive e_Lth root of unity and let ϕ be an L-automorphism of $\mathscr{V}_L(G)$. Then $\phi(\zeta)=\zeta$.*

The proof of Theorem 1 involves an explicit computation using the description of $\mathscr{V}_Q(G)$ given by Amitsur in [2], where Q denotes the rational field.

Suppose G is a finite subgroup of the K-division ring D. Then $C_D(\mathscr{Z}_K)\simeq \mathscr{V}_K(G) \otimes_{\mathscr{Z}_K} A$ where A is a $\mathscr{Z}_K$-division ring and $C_D(\mathscr{Z}_K)$ denotes the centralizer in D of $\mathscr{Z}_K$. Suppose $\zeta \notin K$ where ζ is a primitive e_Kth root of unity. Then there is an automorphism ϕ of $C_D(\mathscr{Z}_K)$ with $\phi(\zeta)\neq\zeta$.

AMS (MOS) subject classifications (1970). Primary 16A40; Secondary 16A72.

[1] Research supported in part by NSF Grant GP-29068.
[2] Research supported in part by NSF Grant GP-28696.

In view of Theorem 1 we are led to consider whether $C_D(\mathscr{Z}_K)$ may be assumed to be $\mathscr{V}_K(G)$.

DEFINITION. Let L be a finite algebraic extension of K and let D be an L-division ring which is a subalgebra of a K-division ring B. We say that D is *maximally embedded* in B if $C_B(L)=D$.

This definition raises the following question:

Question 1. Suppose L is a finite algebraic extension of K, and D is an L-division ring which is a subalgebra of some K-division ring D_0. Does there exist a K-division ring in which D is maximally embedded?

The answer to Question 1 is affirmative if D_0 has its exponent equal to its index (which occurs, for example, if K is a local or global field). We have:

THEOREM 2. *Suppose L is a finite algebraic extension of K and let D be an L-division ring which is a subalgebra of a K-division ring D_0 having equal exponent and index. Then there exists a K-division ring B having equal exponent and index in which D is maximally embedded.*

Combining Theorems 1 and 2 we obtain:

THEOREM 3. *Let D be a K-division ring and let G be a finite subgroup of D^*. Suppose the exponent of D equals the index of D. Then K contains a primitive e_Kth root of unity. In particular, if K contains no nonidentity odd order roots of unity and D is a K-division ring having its exponent equal to its index, then all odd order subgroups of D^* are cyclic.*

Returning to the general situation for arbitrary D, we have $C_D(\mathscr{Z}_K)\simeq \mathscr{V}_K(G)\otimes_{\mathscr{Z}_K} A$. If the exponents of A and $\mathscr{V}_K(G)$ are relatively prime, then the automorphism ϕ of $C_D(\mathscr{Z}_K)$ can be modified to yield an automorphism of $\mathscr{V}_K(G)$ with $\phi(\zeta)\neq\zeta$. This contradiction shows:

THEOREM 4. *The conjecture is true if there are no odd primes whose squares divide the index of D.*

Attempts to modify the automorphism ϕ above lead to the consideration of the following question:

Question 2. Let D be an E-division ring and let ψ be an automorphism of E. Can ψ be extended either to D or to a maximal subfield of D?

The answer to Question 2 is affirmative if E is a local or global field. Moreover, we have:

THEOREM 5. *An affirmative answer to Question* 2 *proves the conjecture.*

Let n be the index of $\mathscr{V}_Q(G)$. Then n is also the index of $\mathscr{V}_L(G)$ for any $L\subset K$ where G is a subgroup of the K-division ring D. Since $\mathscr{Z}_Q$ is an algebraic number field, $e_Q=n$. This leads to our next question:

Question 3. With notation as above, does $e_L=n$?

With regard to Question 3, it should be noted that Albert has given an example in [**1**] of a quadratic extension L of a field K and a K-division ring D of exponent and index 4 such that $D \otimes_K L$ is a division ring of exponent 2.

Finally, we mention that the conjecture is false if the requirement that D be finite dimensional over its center is dropped. Using results of P. M. Cohn we have:

THEOREM 6. *Let G be a finite group which is a subgroup of some division ring. Then G is a subgroup of the multiplicative group of a division ring with center Q. However, if K is an algebraic number field containing no non-identity odd order roots of unity, then G cannot be a subgroup of the multiplicative group of a locally finite division ring with center K.*

Theorem 6 leads us to our final question:

Question 4. Let G be a noncyclic odd order subgroup of some division ring. Does there exist an algebraic division ring with center Q containing G?

Many of the results announced in this paper were obtained at the Research Symposium in Ring Theory held at the University of Chicago during July 1973. The authors would like to express their appreciation of the hospitality shown by the University of Chicago during this period. The details of these results will appear elsewhere.

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