

A NOTE ON ANOSOV DIFFEOMORPHISMS

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1. Introduction. In this note we shall study a class Γ of diffeomorphisms on a compact n -dimensional manifold M . The class Γ will include all diffeomorphisms F with the property that the periodic points of F are dense in M . Our main theorem will give a characterization of those diffeomorphisms in Γ that are Anosov diffeomorphisms.

2. Statement of results. Let $F: M \rightarrow M$ be a diffeomorphism on a compact n -dimensional manifold M and let $DF: TM \rightarrow TM$ be the induced derivative mapping on the tangent bundle of M . The mapping F is said to be an *Anosov diffeomorphism* if the tangent bundle can be decomposed into a continuous Whitney sum $TM = E^s + E^u$, such that

(i) E^s and E^u are invariant under DF ;

(ii) $DF: E^s \rightarrow E^s$ is contracting, i.e., there exist positive constants K and λ , $\lambda < 1$, such that

$$(1) \quad \|DF^m(v)\| \leq K\lambda^m\|v\|$$

for all $v \in E^s$ and $m \in \mathbb{Z}^+$;

(iii) $DF: E^u \rightarrow E^u$ is expanding, i.e., there exist positive constants k and μ , $\mu > 1$, such that

$$(2) \quad \|DF^m(v)\| \geq k\mu^m\|v\|$$

for all $v \in E^u$ and $m \in \mathbb{Z}^+$, cf. [1], [3], and [6].

Since DF is a homeomorphism, the composed mapping DF^m is defined for all $m \in \mathbb{Z}$, and this defines a discrete flow on TM . Similarly F^m is a discrete flow on M , and these flows commute with the natural projection $p: TM \rightarrow M$. Now let Γ denote the collection of all diffeomorphisms $F: M \rightarrow M$ such that the union of the minimal sets of the flow F^m is dense in M . For example, if the periodic points of F are dense in M , then $F \in \Gamma$.

For any diffeomorphism $F: M \rightarrow M$ we define the sets \mathcal{B} , \mathcal{S} , \mathcal{U} in

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the tangent bundle TM by

$$\begin{aligned} \mathcal{B} &= \{v \in TM: \|DF^m(v)\| \text{ is bounded uniformly for } m \in \mathbb{Z}\}, \\ \mathcal{S} &= \{v \in TM: \|DF^m(v)\| \rightarrow 0 \text{ as } m \rightarrow +\infty\}, \\ \mathcal{U} &= \{v \in TM: \|DF^m(v)\| \rightarrow 0 \text{ as } m \rightarrow -\infty\}. \end{aligned}$$

In the theorem, which we next state, the sets \mathcal{S} and \mathcal{U} will take the role of E^s and E^u . However we want to emphasize that the contracting and expanding properties described by (1) and (2) are *not* included in the definition of \mathcal{S} and \mathcal{U} ; they follow as one of the consequences of our theorem.

THEOREM. *Let $F: M \rightarrow M$ be a diffeomorphism on a compact n -dimensional manifold M and assume that $F \in \Gamma$. Then F is an Anosov diffeomorphism if and only if $\mathcal{B} = TM_0$, the zero section of TM . Moreover, in this case \mathcal{S} and \mathcal{U} are subbundles of TM which are invariant under the flow DF^m , $TM = \mathcal{S} + \mathcal{U}$ (Whitney sum), $DF: \mathcal{S} \rightarrow \mathcal{S}$ is contracting, and $DF: \mathcal{U} \rightarrow \mathcal{U}$ is expanding.*

3. Outline of proofs. The proof of this theorem is included in a paper in which we study the general problem of (discrete and continuous) linear flows on vector bundles [4]. The results in this paper also have important applications in the theory of linear differential equations with almost periodic coefficients.

The proof of the necessity of the condition $\mathcal{B} = TM_0$ follows directly from the definition of an Anosov diffeomorphism. The proof of the sufficiency of this condition is accomplished as follows: For $y \in M$ define the fibers

$$\mathcal{S}(y) = p^{-1}(y) \cap \mathcal{S}, \quad \mathcal{U}(y) = p^{-1}(y) \cap \mathcal{U},$$

and define

$$\begin{aligned} A^+ &= \{v \in \mathcal{S}: \|DF^m(v)\| \leq 1 \text{ for all } m \in \mathbb{Z}^+\}, \\ A^- &= \{v \in \mathcal{U}: \|DF^{-m}(v)\| \leq 1 \text{ for all } m \in \mathbb{Z}^+\}. \end{aligned}$$

Note that $\mathcal{S}(y)$ and $\mathcal{U}(y)$ are linear subspaces of T_yM , the tangent space of M at y .

Step 1. A^+ and A^- are compact sets in TM and there is a σ , $0 < \sigma < 1$, such that $\{v \in \mathcal{S}: \|v\| \leq \sigma\} \subseteq A^+$, and $\{v \in \mathcal{U}: \|v\| \leq \sigma\} \subseteq A^-$.

Step 2. \mathcal{S} and \mathcal{U} are closed sets and $\|DF^m(v)\| \leq \sigma^{-1}\|v\|$ for all $v \in \mathcal{S}$ and $m \in \mathbb{Z}^+$, and $\|DF^{-m}(v)\| \leq \sigma^{-1}\|v\|$ for all $v \in \mathcal{U}$ and $m \in \mathbb{Z}^+$.

Step 3. The functions $\dim \mathcal{S}(y)$ and $\dim \mathcal{U}(y)$ are upper semicontinuous functions of y .

Step 4. \mathcal{S} and \mathcal{U} are invariant under DF and $DF: \mathcal{S} \rightarrow \mathcal{S}$ is contracting and $DF: \mathcal{U} \rightarrow \mathcal{U}$ is expanding.

Step 5. If y belongs to a minimal set for the flow F^m on M , then

$$(3) \quad \mathcal{S}(y) + \mathcal{U}(y) = T_y M.$$

The proof of Step 5 uses essentially a duality argument and reduces to examining properties of the intersection number of singular chains in R^n .

Since (3) holds over a dense set in M we show, using Steps 2 and 3, that \mathcal{S} and \mathcal{U} are subbundles and that $TM = \mathcal{S} + \mathcal{U}$ (Whitney sum).

REMARKS. (1) By using a known property of Anosov diffeomorphisms [7], the condition $\mathcal{B} = TM_0$, for $F \in \Gamma$, then implies that the periodic points of F form a dense set in M . Also see [3, p. 116].

(2) If one does not assume that the set $N = \{\text{union of the minimal sets of the flow } F^m\}$ is dense in the manifold M , then it still follows, using our general theory [4], that one gets a corresponding splitting of the tangent bundle over the closure \bar{N} .

We show in [5] that this splitting can be extended to all of M provided the dimension of the fiber $\mathcal{S}(y)$ is the same over every minimal set. This fact has also been discovered independently by R. Mañé Ramirez [2] by different techniques.

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