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SIMPLICITY OF CERTAIN GROUPS OF DIFFEOMORPHISMS

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D. Epstein has shown [1] that for quite general groups of homeomorphisms, the commutator subgroup is simple. In particular, let M be a manifold. (In this note, we assume all manifolds are finite dimensional, Hausdorff, class C^∞, and have a countable basis for their topology.) By a C^{r+} mapping (resp. diffeomorphism) we mean a C^r mapping (resp. diffeomorphism) whose rth derivative is Lipschitz. Let $\mathrm{Diff}(M, r)$ (resp. $\mathrm{Diff}(M, r+)$) denote the group of C^r (resp. C^{r+}) diffeomorphisms h of M such that there is an isotropy H_t of h to the identity, and a compact set K such that $H_t(x)=x$ if $x \in M-K$. Epstein showed in [1] that the commutator subgroup of $\mathrm{Diff}(M, r)$ (resp. $\mathrm{Diff}(M, r+)$) is simple. Thurston announced in [4] that $\mathrm{Diff}(M, \infty)$ is simple. Let $n=\dim M$. In this note we announce the following two results.

THEOREM 1. *If* $\infty \geqq r \geqq n+1$, *then* $\mathrm{Diff}(M, r+)$ *is simple.*

THEOREM 2. *If* $\infty \geqq r \geqq n+1$, *then* $F\Gamma_n^{r+}$ *is* $(n+1)$*-connected.*

Here $F\Gamma_n^{r+}$ denotes Haefliger's classifying space for codimension n foliations of class C^{r+}. These two theorems are closely related by results of Thurston (cf. [2], [4]). The case $r=\infty$ of these theorems is due to Thurston [4].

Here we outline a proof of Theorem 1. By Epstein's theorem, it is enough to show $\mathrm{Diff}(M, r+)$ is equal to its own commutator subgroup. A well-known argument shows that it is enough to prove the latter in the case $M=\boldsymbol{R}^n$. Let $A>1$. Let $D_0=\{x \in \boldsymbol{R}^n: -2 \leqq x_j \leqq 2, 1 \leqq j \leqq n\}$. For $1 \leqq i \leqq n$, let

$$D_i = \{x \in \boldsymbol{R}^n: -2 \leqq x_j \leqq 2, 1 \leqq j < i, -2A \leqq x_j \leqq 2A, i \leqq j \leqq n\}.$$

Let $\alpha \in \mathrm{Diff}(\boldsymbol{R}^n, \infty)$ be such that $\alpha(x)=Ax$ if $x \in D_0$. Let ρ be a C^∞ real valued function on $\boldsymbol{R}^n$, with compact support, such that $0 \leqq \rho \leqq 1$, and $\rho=1$ on D_1. Let $\tau_i=\exp(\rho\partial/\partial x_i)$. Then $\tau_i \in \mathrm{Diff}(\boldsymbol{R}^n, \infty)$, $1 \leqq i \leqq n$.

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LEMMA. *There exists A_0 such that the following holds. Let $A \geqq A_0$. Let f be a C^{r+} diffeomorphism of $\boldsymbol{R}^n$, with support in D_0, and sufficiently close to the identity (with respect to the C^{r+} topology). Suppose $\infty > r \geqq n+1$. Then there exist $g_0, g_1, \cdots, g_n, \lambda_1, \cdots, \lambda_n \in \mathrm{Diff}(\boldsymbol{R}^n, r+)$ such that*

$$\alpha f g_n \alpha^{-1} = g_0, \tag{1}$$

$$\lambda_i g_{i-1} \tau_i \lambda_i^{-1} = g_i \tau_i, \qquad 1 \leqq i \leqq n. \tag{2_i}$$

PROOF THAT THE LEMMA IMPLIES THEOREM 1. It is enough to show that any diffeomorphism such as f is a product of commutators, since any element of $\mathrm{Diff}(\boldsymbol{R}^n, r+)$ is a product of conjugates of such diffeomorphisms. Now if $u \in \mathrm{Diff}(\boldsymbol{R}^n, r+)$, let $[u]$ denote its image in the commutator quotient group. From (1), we get $[f][g_n]=[g_0]$, and from (2_i), we get $[g_{i-1}]=[g_i]$, $1 \leqq i \leqq n$. Hence $[f]=1$. Q.E.D.

OUTLINE OF THE PROOF OF THE LEMMA IN THE CASE $r<\infty$. Let B_δ denote the subset of $\mathrm{Diff}(\boldsymbol{R}^n, r+)$ consisting of g with support in D_0 such that

$$\sup_{x \neq y} \|D^r g(x) - D^r g(y)\| / \|x-y\| < \delta.$$

For $\delta > 0$ sufficiently small, and f sufficiently near the identity, there exists a mapping $\Phi: B_\delta \to B_\delta$ such that if $g \in B_\delta$ and $g_n = \Phi(g)$, then there exist $g_1, \cdots, g_n, \lambda_1, \cdots, \lambda_n \in \mathrm{Diff}(\boldsymbol{R}^n, r+)$ such that

$$\alpha f g \alpha^{-1} = g_0, \tag{3}$$

$$\lambda_i g_{i-1} \tau_i \lambda_i^{-1} = g_i \tau_i, \qquad 1 \leqq i \leqq n. \tag{4_i}$$

The mapping Φ is continuous with respect to the C^r topology. Since B_δ is compact with respect to the C^r topology, and convex, it follows from the Schauder-Tychonoff fixed point theorem that Φ has a fixed point. But such a fixed point provides a solution of the equations (1), (2_i).

We can only sketch the idea of the construction of Φ. If $u: \boldsymbol{R}^n \to \boldsymbol{R}^n$ vanishes outside a compact set, we define

$$\|u\|_{r+} = \sup_{x+y} \|D^r u(x) - D^r u(y)\| / \|x-y\|.$$

Then it is easy to see that if g_0 is defined by (3), we have

$$\|g_0 - \mathrm{id}\|_{r+} < A^{-r} \|fg - \mathrm{id}\|_{r+}.$$

Then we construct $g_1, \cdots, g_n$ inductively. Supposing g_{i-1} has been defined, has support in D_i, and is near the identity, we construct g_i to have support in D_{i+1}, to be near the identity and to satisfy

$$\|g_i - \mathrm{id}\|_{r+} \leqq CA \|g_{i-1} - \mathrm{id}\|_{r+}.$$

Here, C is a constant independent of A. By taking A sufficiently large, and f sufficiently near the identity in relation to δ, we have that Φ maps B_δ into itself, where we define $\Phi(g)=g_n$.

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