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A DYNAMICAL SYSTEM ON E^4 NEITHER ISOMORPHIC NOR EQUIVALENT TO A DIFFERENTIAL SYSTEM

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ABSTRACT. We note that a certain dynamical system on E^4 has local sections which are not classical 3-manifolds. This dynamical system cannot be isomorphic or geometrically equivalent to a differential system on E^4.

Problems 8 and 9 of [3, p. 225] raise the question whether each dynamical system defined on a differentiable manifold is isomorphic or topologically equivalent to a differential system. The purpose of this note is to supply a dynamical system on E^4 which gives a negative answer to the above questions.

DEFINITIONS. A dynamical system on a topological space X is a triple (X, E, π) where E=real number line and $\pi: X\times E\to X$ is a continuous map with the properties that for each $x \in X$, t_1, $t_2 \in E$, $\pi(x, 0)=x$ and $\pi(\pi(x, t_1), t_2)=\pi(x, t_1+t_2)$. A trajectory of (X, E, π) is a set $\pi(\{x\}\times E)$ for a fixed $x \in X$. A rest point of (X, E, π) is a point in X which is also a trajectory.

A local section of extent $\varepsilon>0$ for (X, E, π) is a subset $S\subset X$ with the property that the restriction of π to $S\times(-\varepsilon, \varepsilon)$ is a topological embedding into X. S generates neighborhoods for $K\subset X$ if, for every $\delta>0$, K is interior to $\pi(S\times(-\delta, \delta))$. If S is a local section of extent $\varepsilon>0$, we write $S\pi(-\delta, \delta)$ for $\pi(S\times(-\delta, \delta))$, $0<\delta<\varepsilon$.

THEOREM 1. *Let X be a T_2 topological space, and (X, E, π) a dynamical system on X. Suppose that S and T are each locally compact local sections of extent $\varepsilon>0$ which generate neighborhoods for a point $p \in X$. Then there are relatively open subsets $U\subset S$, $V\subset T$, each containing p, with U homeomorphic to V.*

PROOF. For any space Y, let P_R denote the projection mapping of $Y\times(-\varepsilon, \varepsilon)$ onto $(-\varepsilon, \varepsilon)$. Because $\pi: S\times(-\varepsilon, \varepsilon)\to S\pi(-\varepsilon, \varepsilon)$ is a homeomorphism, $s(x)\equiv P_R\circ\pi^{-1}(x)$ is a continuous map from $S\pi(-\varepsilon, \varepsilon)$ to

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$(-\varepsilon, \varepsilon)$. For any $x \in S\pi(-\varepsilon, \varepsilon)$, $y=\pi(x, -s(x))$ is the unique point in S which belongs to that trajectory segment of $S\pi(-\varepsilon, \varepsilon)$ containing x. Similarly, there is a continuous map $t: T\pi(-\varepsilon, \varepsilon) \to (-\varepsilon, \varepsilon)$ such that for any $v \in T\pi(-\varepsilon, \varepsilon)$, $\pi(v, -t(v))$ is the unique point of T which belongs to that trajectory segment of $T\pi(-\varepsilon, \varepsilon)$ containing v.

Let M be a compact subset of $T \cap (S\pi(-\varepsilon, \varepsilon))$ which contains p on its interior relative to T. Since $M \subset S\pi(-\varepsilon, \varepsilon)$, the map $F: M \to S$ defined by $F(x)=\pi(x, -s(x))$ makes sense and is continuous. For any pair $a, b \in M$, $|s(a)| < \varepsilon$ and $|s(b)| < \varepsilon$. Because the restriction of π to $T \times (-\varepsilon, \varepsilon)$ is injective, and $a, b \in T$, if $\pi(a, -s(a))=\pi(b, -s(b))$ then $a=b$. We see that $F: M \to S$ is injective, and hence a homeomorphism onto its image.

It remains to show that $F(M)$ is a neighborhood of p in S. If there were a net $\{p_\alpha\}$ in $T\pi(-\varepsilon, \varepsilon) \cap (S \setminus F(M))$ converging to p, eventually the net $\{\pi(p_\alpha, -t(p_\alpha))\}$ in T would be in M because this net converges to p in T. Each p_α is in S, and $|t(p_\alpha)| < \varepsilon$, so $F(\pi(p_\alpha, -t(p_\alpha)))=p_\alpha$. A contradiction has been reached, as $\pi(p_\alpha, -t(p_\alpha))$ must eventually be in M, and yet $F(\pi(p_\alpha, -t(p_\alpha)))=p_\alpha$ can never be in $F(M)$. We choose V to be any open subset of M containing p, and set $U=F(V)$. Q.E.D.

Definition. Two dynamical systems (X, E, π) and $(Y, E, \bar{\pi})$ are isomorphic if and only if there is a homeomorphism $f: X \Rightarrow Y$ such that for every $(y, t) \in Y \times E$, $\bar{\pi}(y, t)=f(\pi(f^{-1}(y), t))$.

Lemma 1. *If (X, E, π) and $(Y, E, \bar{\pi})$ are isomorphic dynamical systems and S is a local section of extent ε for (X, E, π), then $f(S)$ is a local section of extent ε for $(Y, E, \bar{\pi})$.*

Proof. The following diagram commutes.

$$\begin{array}{ccc} S \times (-\varepsilon, \varepsilon) & \xrightarrow{\quad \pi \quad} & \pi(S \times (-\varepsilon, \varepsilon)) \\ \Big\uparrow f^{-1} \times \mathrm{id} & & \Big\downarrow f \\ f(S) \times (-\varepsilon, \varepsilon) & \xrightarrow{\quad \bar{\pi} \quad} & \bar{\pi}(f(S) \times (-\varepsilon, \varepsilon)). \end{array}$$

Q.E.D.

Definition. Let $H_0(E, E)$ be the space of homeomorphisms from E onto E which take zero to zero, with the compact-open topology. Two dynamical systems (X, E, π) and $(X, E, \bar{\pi})$ are geometrically equivalent if and only if there is a map $h: X \to H_0(E, E)$ which is continuous except possibly at the rest points of (X, E, π), such that $\bar{\pi}(x, t)=\pi(x, h_x(t))$.

Lemma 2. *If (X, E, π) and $(X, E, \bar{\pi})$ are geometrically equivalent dynamical systems and S is a compact local section of positive extent for (X, E, π), then S is also a local section of positive extent for $(X, E, \bar{\pi})$.*

Proof. With $\delta \equiv \min \max_{x \in S}\{t \in E: |h_x(t)|, |h_x(-t)| \leqq \varepsilon/2\}$, δ is positive because S is compact and the restriction of h to S is continuous. Let e

be the evaluation map $e: H_0(E, E) \times E \rightarrow E$ and note that $\mathrm{id} \times e \circ h$ maps (x, t) to $(x, h_x(t))$. The restriction of $\bar{\pi}$ to $S \times [-\delta, \delta]$ is defined by:

$$\begin{array}{ccc} S \times [-\delta, \delta] & \xrightarrow{\ \bar{\pi}\ } & S\pi(-\varepsilon, \varepsilon) \\ & {\scriptstyle \mathrm{id} \times e \circ h} \searrow \quad \nearrow {\scriptstyle \pi} & \\ & S \times (-\varepsilon, \varepsilon) & \end{array}$$

It is clear that the restriction of $\mathrm{id} \times e \circ h$ to $S \times [-\delta, \delta]$ is an embedding into $S \times (-\varepsilon, \varepsilon)$, so S is a local section of extent δ under $(X, E, \bar{\pi})$. Q.E.D.

Differential systems. Given a locally lipschitzian function $f: E^N \rightarrow E^N$ we may define $\phi(t, y)$ to be the solution, at time t, to the equation $\dot{x} = f(x)$ with the condition $\phi(0, y) = y$. Then $\pi(x, t) = \phi(t, x)$ is a dynamical system map on E^N, since the solutions to $\dot{x} = f(x)$ depend continuously on initial data. We call a dynamical system arising in this way a differential system. It is easy to see that each nonrest point of a differential system on E^N has an $(N-1)$-cell local section that generates neighborhoods for it [**4**, pp. 37–46]. Theorem 1, together with Lemmas 1 and 2, implies that if (E^N, E, π) is a dynamical system which is either isomorphic or geometrically equivalent to a differential system, then each local section of (E^N, E, π) which generates neighborhoods for itself must be a classical $(N-1)$-manifold.

The example. Consider the example, due to Bing, of a nonmanifold $B \subset E^4$ such that there is a homeomorphism $h: B \times E \Rightarrow E^4$. In [**2**, Theorem 13] B is shown to have a cantor set of points where it is a nonmanifold, and in [**1**] the construction of h is accomplished. Let P_B, P_R be the respective projections of $B \times R$ onto B and onto R. We write the arguments of h as pairs in $B \times E$.

Let (E^4, E, π) be the system defined by

$$\pi(x, t) = h(P_B \circ h^{-1}(x), P_R \circ h^{-1}(x) + t).$$

THEOREM 2. *(E^4, E, π) is a dynamical system, and $S \equiv h(B \times \{0\})$ is a local section which generates neighborhoods for itself.*

PROOF. The proof is an easily accomplished verification.

COROLLARY. *(E^4, E, π) is neither isomorphic to nor geometrically equivalent to a differential system on E^4.*

PROOF. The set S defined above is not a classical 3-manifold.

Question. Characterize topologically the dynamical systems which are differential systems.

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