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EXISTENCE OF SOLUTIONS OF DIFFERENTIAL EQUATIONS IN BANACH SPACE

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The results announced here concern the existence of a solution to the general initial value problem

$$(1) \qquad x'(t) = f(t, x(t)), \qquad x(0) = x_0,$$

in which $x(t)$ lies in a Banach space X for $t \in J = [0, a]$. Recent results for this problem have been announced in this Bulletin by S. N. Chow and J. D. Schuur [**1**] and by W. E. Fitzgibbon [**2**]. Related results were obtained earlier by F. Browder [**3**]. Here however, X is not assumed to be separable or reflexive, although as usual f will be continuous in x with respect to the weak topology on X.

A *pseudo-solution* of (1) is an absolutely continuous function $x:J \to X$ with pseudo-derivative (see Pettis [**4**]) satisfying (1). A *strong solution* of (1) is a strongly absolutely continuous function $x:J \to X$ with strong derivative ($\lim_{h\to 0}(x(t+h)-x(t))/h$ in norm) satisfying (1) a.e. on J. For notions of absolute continuity, see Hille and Phillips [**5**, p. 76].

In what follows let B denote an open ball about some point $x_0 \in X$, let $I = [0, b]$ be a compact interval, and let f be a function from $I \times B$ into X.

THEOREM A. *Assume these hypotheses*:

(a) *For a.e.* $t \in I$, $f(t, x)$ *is continuous in the variable* x *with respect to the weak topology on* B *and* X.

(b) *For each strongly absolutely continuous function* $y: I \to B$, $f(t, y(t))$ *is Pettis integrable on* I.

(c) *For some null set* $N \subset I$, *the weak closure of* $f((I-N) \times B)$ *is weakly compact in* X.

Then (1) *has a* (*possibly nonunique*) *pseudo-solution on a subinterval* $J = [0, a]$ *of* I.

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The proof of Theorem A applies the Schauder-Tychonoff fixed point theorem to the transformation T defined by

$$Ty(t) = x_0 + \int_0^t f(s, y(s))\, ds \quad \text{(Pettis integral)}$$

on the intersection of certain convex subsets of the locally convex product space X^J of all functions from J into X.

Sufficient conditions for (b) to hold are that X be weakly sequentially complete, that $f(t, y(t))$ be weakly measurable for each strongly absolutely continuous function $y: I \to B$, and that (c) hold. If we require that $f(t, y(t))$ be strongly measurable, we obtain the existence of a strong solution.

COROLLARY B. *In Theorem* A, *replace condition* (b) *by the following condition.*

(b*) *For every strongly absolutely continuous function* $y: I \to B$, $f(t, y(t))$ *is strongly measurable on* I.

(*This condition, together with* (c), *implies* (b).)

Then every pseudo-solution of (1) *is in fact a strong solution.*

A simple sufficient condition for (b*) to hold is that for each point $x \in B$, $f(t, x)$ be strongly measurable with respect to t on I.

Strong solutions of (1) will also be obtained in Theorem A if X is uniformly convex, for pseudo-solutions of (1) under hypothesis (c) are in fact strongly absolutely continuous, hence strongly differentiable by Clarkson [**6**].

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