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QUASI-KAN EXTENSIONS FOR 2-CATEGORIES

BY JOHN W. GRAY[1]

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1. **Introduction.** Let $\mathscr{C}at$ denote the category of small categories and functors. $\mathscr{C}at$ is a Cartesian closed category, [**2**] and the prefix 2- will denote categories and functors enriched in $\mathscr{C}at$. 2-$\mathscr{C}at$ denotes the category of small 2-categories and 2-functors. It is also Cartesian closed, but there is another notion of a transformation between 2-functors F and G which has interesting properties; namely a *quasi-natural* transformation from F to G is a family of morphisms $\{\varphi_A : F(A) \to G(A)\}$ together with a family of 2-cells $\{\varphi_f : G(f)\varphi_A \to \varphi_B F(f)\}$ as illustrated

$$\begin{array}{ccc} F(A) & \xrightarrow{F(f)} & F(B) \\ {\scriptstyle\varphi_A}\downarrow & \quad\varphi_f \nearrow & \downarrow{\scriptstyle\varphi_B} \\ G(A) & \xrightarrow[G(f)]{} & G(B) \end{array}$$

satisfying obvious compatibility conditions. (The case where the φ_f's are isomorphisms has been considered in [**7**] and [**8**], but we make no such restriction.) Given this notion of "natural transformation", it is reasonable and useful to inquire about the corresponding notion of "quasi-limit" or, more generally, "quasi-Kan extension".

Such a Kan extension was used in an essential way for the proof of the main result in [**4**, §9], but until now no justification has been given for calling the construction used there a "Kan extension". In the usual case, if $S : \mathscr{A} \to \mathscr{B}$ is an ordinary functor and $\mathscr{X}$ is a cocomplete category, then under appropriate hypotheses the functor

$$\mathscr{X}^S : \mathscr{X}^B \to \mathscr{X}^A$$

is right adjoint to the (left) Kan extension $\Sigma S : \mathscr{X}^A \to \mathscr{X}^B$. ΣS can be constructed as follows: replace S by its associated factorization through an opfibration

$$\mathscr{A} \underset{P}{\overset{Q_S}{\rightleftarrows}} (S, \mathscr{B}) \xrightarrow{P_S} \mathscr{B}$$

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where $S=P_S Q_S$ and P is left inverse, right adjoint to Q_S (see [**3**, p. 55]). One shows that for the opfibration P_S, the Kan extension ΣP_S is given by "integration (i.e., colimit) along the fibres" and then that $\Sigma S=(\Sigma P_S)\mathscr{X}^P$.

In the 2-category case, 2-functors and quasi-natural transformations are the objects and morphisms of a 2-category Fun($\mathscr{A}$, $\mathscr{B}$) which is the internal hom object for a nonsymmetrical, monoidal closed structure on 2-$\mathscr{C}at$, denoted by 2-$\mathscr{C}at_{\otimes}$. (Cf. [**4**, p. 280] and [**6**].) If $S:\mathscr{A}\to\mathscr{B}$ is a fixed 2-functor, then for any $\mathscr{X}$, there is an induced functor

$$S^* = \mathrm{Fun}(S, \mathscr{X}):\mathrm{Fun}(\mathscr{B}, \mathscr{X}) \to \mathrm{Fun}(\mathscr{A}, \mathscr{X})$$

and we can ask for a left quasi-adjoint $\Sigma_q S$ to S^* in some suitable sense of quasi-adjointness. (The case $\mathscr{B}=1$ yields quasi-colimits.) In this paper we describe how to modify each step in the procedure described above to fit the situation of 2-categories. It will be seen that when S is a suitable kind of quasi-opfibration, one obtains an ordinary $\mathscr{C}at$-enriched adjoint ($=\mathscr{C}at$-adjoint). This includes the case of quasi-limits. However, in general one gets a *strict quasi-adjoint*. This notion forces itself upon one when one studies 2-$\mathscr{C}at_{\otimes}$ seriously, since it arises in many different contexts. (In particular, the comprehension scheme in [**4**] is a strict quasi-adjoint, as are all the constructions mentioned in this paper.) Detailed proofs will be published in [**6**].

2. **Definitions.** Besides the 2-comma category $[S_1, S_2]$ defined as in [**4**, p. 279], for a pair of 2-functors $S_i:\mathscr{A}_i\to\mathscr{B}$, there are 3-comma categories $[S_1, S_2]_3$ and $[S_1, S_2]_{\otimes}$ defined for 3-functors (resp. 2-$\mathscr{C}at_{\otimes}$-functors) S_1 and S_2 between 3-categories (resp., 2-$\mathscr{C}at_{\otimes}$-categories) with the same codomain. 0-cells and 1-cells are defined as in $[S_1, S_2]$, 2-cells are a pair of 2-cells as in $[S_1, S_2]$ plus a 3-cell expressing the lack of commutativity, and 3-cells are pairs satisfying the obvious equation. Details will be given in [**6**].

Let $F:\mathscr{A}\to\mathscr{B}$ and $U:\mathscr{B}\to\mathscr{A}$ be 2-functors between 2-categories. A pair of quasi-natural transformations, $\varepsilon:FU\to\mathscr{B}$ and $\eta:\mathscr{A}\to UF$ is called a *quasi-adjunction* between F and U if it satisfies the usual equations. It is called *strict* if

$$(U\varepsilon F)(\eta_\eta) = 1_\eta, \qquad \varepsilon_\varepsilon(F\eta U) = 1_\varepsilon.$$

Here, for instance, since $\eta_A:A\to UFA$ is an arrow, η assigns to it a 2-cell from $(UF\eta_A)\eta_A$ to $(\eta UF_A)\eta_A$. This defines the modification (2-cell in Fun($\mathscr{A}$, $\mathscr{A}$)) denoted by η_η. Similarly, 1_η is the identity modification of η.

3. **Cartesian quasi-limits and colimits.** In the special case of induced functors S^* in which $\mathscr{B}=\mathbf{1}$, one obtains the constant embedding

$$\Delta_{\mathscr{X}}:\mathscr{X} \to \mathrm{Fun}(\mathscr{A}, \mathscr{X}).$$

The right (resp. left) $\mathscr{C}at$-adjoint to $\Delta_{\mathscr{X}}$ is called the *Cartesian quasi-limit* (resp., *colimit*) of type $\mathscr{A}$ in $\mathscr{X}$ and is denoted by

$$\underset{\rightarrow}{Q}_{\mathscr{A}} \underset{\mathscr{C}at}{\dashv} \Delta_{\mathscr{X}} \underset{\mathscr{C}at}{\dashv} \underset{\leftarrow}{Q}_{\mathscr{A}}$$

$\mathscr{X}$ is called *Cartesian quasi-complete* (resp., *cocomplete*) if $\underset{\rightarrow}{Q}_{\mathscr{A}}$ (resp., $\underset{\leftarrow}{Q}_{\mathscr{A}}$) exists in $\mathscr{X}$ for all small 2-categories $\mathscr{A}$.

THEOREM 1, *$\mathscr{C}at$ is Cartesian quasi-complete and cocomplete.*

PROOF. We describe the construction here.

Let $H:\mathscr{A}\to\mathscr{C}at$ be a 2-functor. H determines an opfibration $P:\mathscr{E}_H\to\mathscr{A}$ (see [**4**, pp. 248, 285, 289]) where $\mathscr{E}_H$ is a 2-category. In the special case that $H:\mathscr{A}\to\mathscr{S}ets$ with $\mathscr{A}$ an ordinary category, then $\varinjlim H=\pi_0(\mathscr{E}_H)$ and $\varprojlim H=\Gamma(\mathscr{E}_H)$ where π_0 assigns to a category its set of connected components, and Γ is the set of sections of P. In the general case one shows that $\underset{\rightarrow}{Q}H=L\pi_0(\mathscr{E}_H)$ and $\underset{\leftarrow}{Q}H=\Gamma(\mathscr{E}_H)$ where $L\pi_0$:2-$\mathscr{C}at\to\mathscr{C}at$ is "local π_0", i.e., it turns a 2-category $\mathscr{C}$ into a category $L\pi_0(\mathscr{C})$ by replacing each hom-category $\mathscr{C}(X, Y)$ by the set $\pi_0\mathscr{C}(X, Y)$. (Note that this differs from the assertion in [**4**, p. 289].) Similarly, Γ denotes the category of sections of P; i.e., 2-functors $G:A\to\mathscr{E}_H$ such that $PG=1$ and natural transformations ψ such that $P\psi=1$.

In [**5**] it is asserted, and it will be proved elsewhere, that this result holds for strongly representable (resp., corepresentable) 2-categories. These are essentially 2-categories which are complete (resp., cocomplete) in the sense of closed categories.

We list here a number of examples of Cartesian quasi-limits and colimits in $\mathscr{C}at$. These examples serve to define the corresponding concepts in other 2-categories.

(i) $\mathscr{A}=\mathbf{2}$ (the category with two objects and a single nonidentity morphism). $H:\mathbf{2}\to\mathscr{C}at$ looks like a functor $f:A\to B$ between small categories and $\underset{\rightarrow}{Q}H=(f, B)$, the universal opfibration associated to f, while $\underset{\rightarrow}{Q}H=\langle A, f\rangle$, the universal cofibration associated to f (see [**3**, §5]).

(ii) (cf. Street [**10**]) Let Δ^{op} denote the 2-category with a single object $*$, with Hom($*$, $*$) the dual of the category of finite ordinals. A 2-functor $H:\mathbf{\Delta}^{\mathrm{op}}\to\mathscr{C}at$ is the same as a small category $\mathscr{A}$ equipped with a cotriple G, and $\underset{\rightarrow}{Q}H$ is the co-Kleisli category of the cotriple. Let ${}^{\mathrm{op}}(\mathbf{\Delta}^{\mathrm{op}})$ be the weak dual. Then $H:{}^{\mathrm{op}}(\mathbf{\Delta}^{\mathrm{op}})\to\mathscr{C}at$ is a small category equipped with a triple and $\underset{\leftarrow}{Q}H$ is the category of Eilenberg-Moore algebras. Appropriate duals give the other two possibilities.

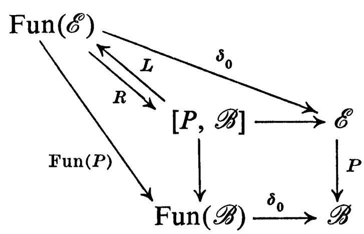
If the possibilities are extended by allowing nonfull subcategories of Fun($\mathscr{A}$, $\mathscr{B}$) determined by imposing conditions on the 2-cells φ_f for certain f's, then $\mathscr{C}at$ still admits such quasi-limits and colimits. As particular examples, one obtains comma categories [**3**] and subequalizers (Lambek [**9**]) as well as the result that the closure of $\mathscr{S}ets \subset \mathscr{C}at$ under such quasi-colimits is all of $\mathscr{C}at$.

The main result about $\underset{\to}{Q}$ needed for quasi-Kan extensions is the following.

THEOREM 2. *$\underset{\to}{Q}:{}_s[2\text{-}\mathscr{C}at_\otimes, \mathscr{X}]_\otimes \to \mathscr{X}$ is a 2-$\mathscr{C}at_\otimes$-functor which is the left 2-$\mathscr{C}at_\otimes$-adjoint to N.*

Here N is the functor in the other direction which is the name functor; e.g., on an object $X \in \mathscr{X}$, $N(X) = X: 1 \to \mathscr{X}$, etc., and s means small. The main (and considerable) difficulty is to show that $\underset{\to}{Q}$ is defined here.

4. **Quasi-fibrations.** Among the various possible definitions the following is the one needed here. Let Fun$(B) = {}^{\mathrm{op}}$Fun(**2**, ${}^{\mathrm{op}}\mathscr{B}$). *A* 2-functor $P: \mathscr{E} \to \mathscr{B}$ is called a *Cartesian quasi-opfibration* if there exists a 2-functor L as illustrated



having R as a right $\mathscr{C}at$-adjoint and $RL = \mathrm{id}$. Here the square is a pullback and δ_0, Fun(P) and R are the obvious induced 2 functors. A choice of L is called a *cleavage*. If L is chosen so that $L(\mathrm{id}) = \mathrm{id}$ and

$$L(f_*E, g) \circ L(E, f) = L(E, gf)$$

then P together with L is called a *split-normal* Cartesian quasi-opfibration. The 2-category of such together with cleavage preserving 2-functors and $\mathscr{C}at$ natural transformations over $\mathscr{B}$ is denoted by Cart q-Split$(\mathscr{B})_0$.

THEOREM 3. *The inclusion*

$$\mathrm{Cart}\ q\text{-Split}(\mathscr{B})_0 \to [{}^{\mathrm{op}}2\ \mathscr{C}at, \mathscr{B}]$$

has a strict left quasi-adjoint Φ.

Here Φ on an object $S: \mathscr{A} \to \mathscr{B}$ is the projection $P_S: [S, \mathscr{B}] \to \mathscr{B}$.

THEOREM 4. *The associated opfibration has the property that there is a factorization of S,*

$$\mathscr{A} \underset{P}{\overset{Q_S}{\rightleftarrows}} [S, \mathscr{B}] \xrightarrow{P_S} \mathscr{B}$$

in which $P_S Q_S = S$ *and P is left inverse, strict quasi-right adjoint to* Q_S.

REMARK. P_S is also an ordinary $\mathscr{C}at$-enriched opfibration and P is a $\mathscr{C}at$-enriched fibration.

THEOREM 5. *Let* $(P:\mathscr{E}\to\mathscr{B}, L)$ *be a split normal Cartesian quasi-opfibration with small fibres. Then there is a* 2-$\mathscr{C}at_\otimes$*-imbedding* $J:\mathscr{B}\to {}_s[2\text{-}\mathscr{C}at_\otimes, \mathscr{E}]_\otimes$.

5. **Quasi-Kan extensions.** For quasi-opfibrations, one has the following astonishing result.

THEOREM 6. *If* $P:\mathscr{E}\to\mathscr{B}$ *is a split normal Cartesian quasi-opfibration with small fibres, and* $\mathscr{X}$ *is Cartesian quasi-cocomplete then* $P^*:\mathrm{Fun}(\mathscr{B}, \mathscr{X})\to \mathrm{Fun}(\mathscr{E}, \mathscr{X})$ *has a left* $\mathscr{C}at$*-adjoint,* $\Sigma_q P$ *given by "integration along the fibres."*

This means that if $G:\mathscr{E}\to\mathscr{X}$ is a 2-functor, then $\Sigma_q P(G)$ is the composition

$$\mathscr{B} \xrightarrow{J} [2\text{-}\mathscr{C}at_\otimes, \mathscr{E}]_\otimes \xrightarrow{G_*} {}_s[2\text{-}\mathscr{C}at_\otimes, \mathscr{X}]_\otimes \xrightarrow{Q} \mathscr{X}$$

where G_* denotes composition with G.

Finally, we get the desired generalization of Kan extensions.

THEOREM 7. *Let* $S:\mathscr{A}\to\mathscr{B}$ *be a* 2*-functor between small* 2*-categories and let* $\mathscr{X}$ *be Cartesian quasi-cocomplete. Let* $\Sigma_q S = (\Sigma_q P_S)P^*$ *where* P_S *and P are as in Theorem* 4. *Then* ${}^{\mathrm{op}}(\Sigma_q S)$ *is a strict quasi-left-adjoint to* ${}^{\mathrm{op}}(S^*)$: ${}^{\mathrm{op}}\mathrm{Fun}(\mathscr{B}, \mathscr{X})\to {}^{\mathrm{op}}\mathrm{Fun}(\mathscr{A}, \mathscr{X})$.

The claim in [**4**, §9], about $\Sigma_q S$ is incorrect and the adjunction is in the sense stated here. Part of the reason for the failure of $\Sigma_q S$ to be a $\mathscr{C}at$-adjoint in general is that P is transversal to the fibres in $[S, \mathscr{B}]$ which has the effect that $\Sigma_q S$ applied to a quasi-natural transformation yields a $\mathscr{C}at$-natural transformation. An example of $\Sigma_q S$ is given in [**4**]. Others will be given elsewhere in the subject of 2-theories.

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Department of Mathematics, University of Illinois at Urbana-Champaign, Urbana, Illinois 61801