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GENERAL INTERPOLATING SEQUENCES IN DISKS AND POLYDISKS

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Let A be a uniform algebra, i.e. a closed subalgebra of the continuous functions on a compact Hausdorf space which separates points, contains constants, and is equipped with the sup norm topology. Let D^n be the (open) unit polydisk in C^n; let $H^\infty(D^n, A)$ be the Banach algebra of bounded, analytic, A-valued functions on D^n equipped with the sup norm (see [3, pp. 224–232] for a description of such functions); and let $l^\infty A$ be the Banach algebra of bounded sequences of elements of A. If $S=\{a_i\}_{i=1}^\infty \in D^n$ is a sequence of points in D^n, we can define a map, $T: H^\infty(D^n, A)\to l^\infty A$ by $T(f)=\{f(a_i)\}_{i=1}^\infty$. We will say S is an interpolating sequence w.r.t. A if T is surjective. If S is an interpolating sequence w.r.t. C we will simply say S is an interpolating sequence, and if S is an interpolating sequence w.r.t. every uniform algebra, we will say S is a general interpolating sequence.

If $S=\{a_i\}_{i=1}^\infty \subset D^n$ is an interpolating sequence, then it is known that S must be uniformly separated, i.e. there exists a constant M and functions $f_1, f_2, \cdots \in H^\infty(D^n)$ such that for all i, $\|f_i\|\leqq M$ and $f_i(a_i)=1$ while f_i is zero on the remaining points of S. (We will use $H^\infty(D^n)$ for $H^\infty(D^n, C)$.) In 1958, L. Carleson [2] showed that for $S\subset D$ $(D=D^1)$, uniform separation is a necessary and sufficient condition that S be an interpolating sequence (w.r.t. C). We will produce a sufficient condition for a sequence to be a general interpolating sequence, and indicate some additional criteria, which, along with uniform separation, guarantees general interpolation.

1. General interpolating, uniformly separated, and α-separated sequences.

If S is an interpolating sequence w.r.t. A, the map $T: H^\infty(D^n, A)\to l^\infty A$

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described above is surjective, and the open mapping theorem implies the existence of a constant M_A such that, for any element $\{X_i\}_{i=1}^{\infty} \in l^{\infty}A$, there exists $f \in H^{\infty}(D^n, A)$ such that $\|f\| \leqq M_A(\mathrm{Sup}\|X_i\|)$ and $f(a_i)=X_i$, $i=1, 2, \cdots$. Any such M_A will be called an interpolating bound for S w.r.t. A.

PROPOSITION 1.1. *If S is a general interpolating sequence, there exists a single constant M such that, for any uniform algebra A, there is an interpolating bound for S w.r.t. A which is less than M.*

DEFINITION 1.2. Any constant M with the property of Proposition 1.1 will be called a general interpolating bound for S.

DEFINITION 1.3. A sequence $S=\{a_i\}_{i=1}^{\infty} \in D^n$ will be called a uniformly separated (u.s.) sequence if there exists a constant $M>0$ and functions $f_1, f_2, \cdots \in H^{\infty}(D^n)$ with the property that for all i, $\|f_i\| \leqq M$ and $f_i(a_i)=1$ while $f_i(a_j)=0$ for $j \neq i$.

We have noted in the introduction that if $S \subset D$, then S is interpolating iff S is u.s. In this case, by means of Blaschke products, we see that S is a u.s. sequence in D iff

$$\operatorname*{Inf}_{i} \prod_{j=1, j\neq i}^{\infty} \left| \frac{a_i - a_j}{1 - a_i \bar{a}_j} \right| > 0.$$

This last condition is satisfied iff $S \subset D$ has the following property:

There exist constants $M>0$, $p>0$ and functions $\theta_1, \theta_2, \cdots \in H^{\infty}(D)$ such that

1. $\|\theta_i\| \leqq 1$, for all i.
2. $\theta_i(a_i)=0$ while $\|\theta_i(a_j)\| > p$ if $i \neq j$.
3. $\sum_{j=1}^{\infty} 1 - |\theta_j(a_i)| \leqq M$ for all i.
4. $\sum_{j=1}^{\infty} |1-\theta_j(z)|$ converges uniformly on compact subsets of D.

The function $\theta_j(z)$ is just $\bar{a}_j(a_j - z)/|a_j|(1-\bar{a}_j z)$.

By strengthening the third requirement in the above property, we define a new property which we will show is sufficient for general interpolation in D^n.

DEFINITION 1.4. A sequence $S=\{a_i\}_{i=1}^{\infty} \subset D^n$ will be called α-separated if there exist constants $m, p>0$ and functions $\alpha_1, \alpha_2, \cdots \in H^{\infty}(D^n)$ such that

1. $\|\alpha_i\| \leqq 1$ for all i.
2. $\alpha_i(a_i)=0$ while $|\alpha_i(a_j)| > p$ for $i \neq j$.
3. $\sum_{j=1}^{\infty} |1-\alpha_j(a_i)| \leqq m$ for all i.
4. $\sum_{j=1}^{\infty} |1-\alpha_j(z)|$ converges uniformly on compact subsets of D^n.

If S is α-separated, and $m, p, \alpha_1, \alpha_2, \cdots$ are as in the definition, then $f_i = \prod_{j=1, j\neq i}^{\infty} \alpha_j$ will be a well-defined element of $H^{\infty}(D^n)$, $\|f_i\| \leqq 1$, and $f_i(a_j)=0$ for $i \neq j$ while $|f_i(a_i)| \geqq e^{-m/p}$ for all i. Consequently S is u.s.

On the other hand it is not true that all u.s. sequences in D^n (for $n>1$) are α-separated. (The situation for $n=1$ is not clear.)

2. Main result.

LEMMA 2.1. *Suppose A, B are Banach spaces, $T\in \mathrm{Hom}(A, B)$, and E is the unit ball of B. Suppose there exist constants $0<k<1$ and $K>0$ such that for all $Y\in B$, there is an $X\in A$ such that $\|TX-Y\|\leqq k$ and $\|X\|\leqq K$, then T is onto, and for all Y in B there exists X in A such that $TX=Y$ and $\|X\|\leqq K/(1-k)$.*

PROOF. See for example Theorem 1.2 of [1] for a generalization.

THEOREM 2.2. *If $S=\{a_i\}_{i=1}^{\infty}\subset D^n$ is α-separated, then S is a general interpolating sequence with bound depending only on the parameters m, p of the definition of α-separation.*

PROOF. Let $m, p, \alpha_1, \alpha_2, \cdots, f_1, f_2, \cdots$ be as in Definition 1.4 and the discussion following it except that we will assume that $f_i(a_i)=\delta=e^{-m/p}$. We first dispense with the following technical detail.

LEMMA 2.3. *If $0<b<1$ then $\prod_{j=1,j\neq i}^{\infty} b\alpha_j(a_i)/(1-(1-b)\alpha_j(a_i))$ converges. Moreover,, if $c_i=\prod_{j=1,j\neq i}^{\infty} b\alpha_j(a_i)/(1-(1-b)\alpha_j(a_i))$, b can be chosen so that $|c_i|\geqq 1/2\delta$ for $i=1, 2, \cdots$.*

PROOF. A straightforward calculation shows that

$$\left|1-\frac{b\alpha_j(a_i)}{1-(1-b)\alpha_j(a_i)}\right| \leqq (1/b)\,|1-\alpha_j(a_i)|$$

so

$$\sum_{j=1}^{\infty}\left|1-\frac{b\alpha_j(a_i)}{1-(1-b)\alpha_j(a_i)}\right| \leqq m/b \quad \text{for all } i.$$

Consequently $\prod_{j=1,j\neq i}^{\infty} b\alpha_j(a_i)/(1-(1-b)\alpha_j(a_i))$ converges for all i. Moreover,

$$\left|\frac{b\alpha_j(a_i)}{1-(1-b)\alpha_j(a_i)}\right| \geqq \frac{bp}{1+(1-b)}=\frac{bp}{2-b} \quad \text{if } i\neq j.$$

Therefore

$$\left|\prod_{j=1,j\neq i}^{\infty}\frac{b\alpha_j(a_i)}{1-(1-b)\alpha_j(a_i)}\right| \geqq \exp\left(-\left(\frac{2-b}{b^2}\right)m/p\right)=\delta^{(2-b)/b^2}.$$

For values of b between 0 and 1, $\delta^{(2-b)/b^2}$ is an increasing function of b with values 0 at $b=0$ and δ at $b=1$. Thus we can find a specific b, $0<b<1$, such that $\delta^{(2-b)/b^2}=1/2\delta$.

Returning to the proof of Theorem 2.2, fix a uniform algebra A, a constant $0<r<1$, and suppose $Y_1, Y_2, \cdots \in A$ such that $\|Y_i\| \leqq r$. Fix b as in Lemma 2.3 and for $j=1, 2, \cdots$ let $\omega_j = f_j Y_j$ and

$$h_j = \frac{\alpha_j\omega_j + b\alpha_j - (1-b)\omega_j}{1 + b\omega_j - (1-b)\alpha_j}.$$

Then $h_j \in H^\infty(D^n, A)$ and $\|h_j\| \leqq 1$. Also, $h_j(a_i) = b\alpha_j(a_i)/(1-(1-b)\alpha_j(a_i))$ for $i \neq j$, so $\prod_{j=1, j\neq i}^\infty h_j(a_i) = c_i$ (where c_i is as in Lemma 2.3). On the other hand, $h_i(a_i) = -(1-b)\delta Y_i/(1+b\delta Y_i)$.

Consequently, if we define $G(z) = \prod_{j=1}^\infty h_j(z)$ and put aside for the moment the question of whether G is convergent, bounded, or analytic, we have $G(a_i) = h_i(a_i)\prod_{j=1, j\neq i}^\infty h_j(a_i) = -(1-b)\delta Y_i c_i/(1+b\delta Y_i)$.

Now suppose $\{X_i\}_{i=1}^\infty \in l^\infty A$. If $Y_i = -5X_i b/2c_i(1-b)$, then

$$G(a_i) = (5/2)b\delta X_i/(1 + b\delta Y_i)$$

and

$$\|X_i - G(a_i)\| = \|X_i\| \left\| \frac{1 - (5/2)b\delta + b\delta Y_i}{1 + b\delta Y_i} \right\|$$
$$\leqq \|X_i\| \frac{(1 - (5/2)b\delta) + b\delta}{1 - b\delta} = \varepsilon_0 \|X_i\|$$

where $\varepsilon_0 = (2-3b\delta)/(2-2b\delta) < 1$. We must of course restrict X_i so that $\|Y_i\| \leqq r$. This can be accomplished by requiring that

$$\|X_i\| \leqq (1-b)\delta r/5b \quad \text{for all } i.$$

Recapitulating, we have constants δ and b depending only on m and p (the parameters of α-separation for S) such that given any element $\{X_i\}_{i=1} \in l^\infty A$, with $\mathrm{Sup}_i \|X_i\| \leqq (1-b)\delta r/5b$, there exists G (which we have not yet shown is in $H^\infty(D^n, A)$) with the property that

$$\|G(a_i) - X_i\| \leqq \varepsilon_0 \|X_i\| \quad \text{for } i = 1, 2, \cdots.$$

If G is in $H^\infty(D^n, A)$, then $\|G\| \leqq 1$, and we could apply Lemma 2.1 to show that the mapping $T: H^\infty(D^n, A) \to l^\infty A$ is onto, and consequently S is interpolating w.r.t. A with bound

$$\frac{5b}{(1-b)\delta r(1-\varepsilon_0)} = \frac{10(1-b\delta)}{(1-b)\delta^2 r} \quad (r \text{ arbitrarily close to } 1).$$

The proof of Theorem 2.2 is complete once we have shown $G \in H^\infty(D^n, A)$. This follows from the following lemma.

LEMMA 2.4. *If $Y_1, Y_2, \cdots \in A$, $\|Y_j\| \leqq r < 1$ for all j, then $\prod_{j=1}^\infty h_j(z)$ converges uniformly on compact polydisks in D^n.*

PROOF. Using the notation in which h_j was originally defined:

$$\|h_j(z) - 1\| = \left\| \frac{(\omega_j(z) + 1)(1 - \alpha_j(z))}{1 + b\omega_j(z) - (1 - b)\alpha_j(z)} \right\|$$
$$\leqq \frac{(\|Y_j\| + 1)\,|1 - \alpha_j(z)|}{1 - b\,\|Y_j\| - (1 - b)}$$
$$\leqq \frac{1}{b}\frac{1 + r}{1 - r}|1 - \alpha_j(z)|\,.$$

So by property (4) of Definition 1.4, $\sum_{j=1}^{\infty} \|h_j(z)-1\|$ converges uniformly on compact polydisks in D^n. By standard arguments, this implies that the partial products, $\prod_{j=1}^{N} h_j(z)$, $N=1, 2, \cdots$, are uniformly Cauchy and hence convergent on compact polydisks in D^n.

3. **Some applications of Theorem 2.2.** It is possible to find many examples of α-separated sequences. One of the most useful examples is given by the following:

PROPOSITION 3.1. *If $S=\{a_i\}_{i=1}^{\infty}\subset D$, is u.s. and there exists $\tau>0$ such that $S\subset\{z\in D:(1-|z|^2)/|1-z^2|>\tau\}\cap\{z\in D:\operatorname{Im}(z)\geqq 0\}$ then S is α-separated.*

This proposition is proved by showing that the functions

$$\alpha_i(z) = \frac{z - a_i}{1 - z\bar{a}_i}\frac{z - \bar{a}_i}{1 - za_i}$$

satisfy the conditions of Definition 1.4.

COROLLARY 3.2. *If $S=\{a_i\}_{i=1}^{\infty}\subset D$ is u.s. and there exists $\tau>0$ such that $S\subset\{z\in D:(1-|z|^2/|1-z^2|>\tau\}$ then S is a general interpolating sequence.*

This last corollary can be generalized to polydisks. To begin with, we recognize that the region $\{z\in D:(1-|z|^2)/|1-z^2|>\tau\}$ is the intersection of two disks whose boundaries intersect one another twice on the boundary of D. We could call the intersection of any two disks whose boundaries intersect one another twice on the boundary of D a wedge. By means of Möbius transformations on D, any such wedge can be mapped into a region like the one in Corollary 3.2. This immediately yields:

COROLLARY 3.3. *If $S\subset D$ is a uniformly separated sequence contained in a wedge, then S is a general interpolating sequence.*

We can define a wedge in D^n as the Cartesian product of n wedges in D, and a near-wedge in D^n as the Cartesian product of one copy of D with $n-1$ wedges in D.

THEOREM 3.4. *If $S \subset D^n$ is a uniformly separated sequence, then*

(1) *If S is contained in a finite union of wedges, it is a general interpolating sequence.*

(2) *If S is contained in a near-wedge, it is an interpolating sequence.*

This theorem is not proved by exhibiting functions which satisfy Definition 1.4, but involves other techniques of constructing uniformly separated and interpolating sequences. For example, case one is obtained by "jiggling" the points of a union of a finite number of Cartesian products of general interpolating sequences in D.

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