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CENTRAL MULTIPLIER THEOREMS FOR COMPACT LIE GROUPS

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The purpose of this note is to describe how central multiplier theorems for compact Lie groups can be reduced to corresponding results on a maximal torus. We shall show that every multiplier theorem for multiple Fourier series gives rise to a corresponding theorem for such groups and, also, for expansions in terms of special functions.

We use the notation and terminology of N. J. Weiss [4]. Let G denote a simply connected semisimple Lie group, $\mathfrak{g}$ its Lie algebra and $\mathfrak{h}$ a maximal abelian subalgebra; P^+ the set of positive roots in $\mathfrak{h}^*$, the dual of $\mathfrak{h}$ (with respect to some order), and $(\ ,\ )$ is the inner product on $\mathfrak{h}^*$ induced by the Killing form. With $\lambda=(\lambda_1, \cdots, \lambda_l) \in \mathbf{Z}^l$ we associate the weight $\lambda=\sum_{i=1}^{l} \lambda_i\pi_i$, where π_i are the fundamental weights adapted to the simple roots. The characters χ_λ of G are then indexed by those λ with nonnegative integer coefficients. The degree d_λ of the corresponding representation is then given by

$$d_\lambda = \prod_{\alpha\in P^+} (\lambda + \beta, \alpha) \Big/ \prod_{\alpha\in P^+} (\beta, \alpha),$$

where $\beta=\frac{1}{2} \sum_{\alpha\in P^+} \alpha$. We now define the difference operator $\mathscr{D}$ on sequences m_λ, $\lambda \in \mathbf{Z}^l$, by first putting $D_\alpha m_\lambda = m_{\lambda-\alpha} - m_\lambda$ (where the root α is identified with its coordinates with respect to the basis of π_i's) and then letting

$$\mathscr{D} m_\lambda = \left(\prod_{\alpha\in P^+} D_\alpha\right) m_\lambda;$$

this is a difference operator of order $(n-l)/2$ ($n=\dim G$, $l=\dim \mathfrak{h}$).

A central convolution operator M on G admits a formal expansion $M \sim \sum_{\lambda_i \geq 0} d_\lambda m_\lambda \chi_\lambda$. The sequence $\{m_\lambda\}$ is called a multiplier for $L^p(G)$ if the operator $M * f = \sum d_\lambda m_\lambda (\chi_\lambda * f)$, defined for generalized trigonometric polynomials f (see [3]), can be extended to a bounded operator on $L^p(G)$.

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The *symmetric trigonometric polynomials* on $\mathfrak{h}/\mathbf{Z}^l$ are defined by $C_\lambda(\tau)=\sum_{\sigma\in W} e^{i(\lambda,\sigma(\tau))}$, where W is the Weyl group. We can now state

THEOREM I. *$\{m_\lambda\}$ defines a bounded operator on $L^p(G)$, $1\leqq p\leqq\infty$, if $\sum_{\lambda_i\geqq 0}\mathscr{D}(d_\lambda m_\lambda)C_\lambda(\tau)$ defines a bounded operator on $L^p(\mathfrak{h}/\mathbf{Z}^l)$. (If one coordinate λ_i is negative, $m_\lambda=0$.) In addition, for $p=1$ the condition is necessary and sufficient. For $1<p<\infty$ it is enough to assume that $\sum\mathscr{D}(d_\lambda m_\lambda)e^{i(\lambda,\tau)}$ defines a bounded operator on the torus.*

We obtain the result of N. J. Weiss on $L^p(G)$ by using Hörmander's multiplier theorem for the torus T^l. The estimates on $\mathscr{D}(d_\lambda m_\lambda)$ can be obtained by observing that $\mathscr{D}$ is a difference operator of order $(n-l)/2$ and d_λ is a polynomial in λ of degree $(n-l)/2$ satisfying the estimate $|d_\lambda|\leqq C|\lambda|^{(n-l)/2-1}$ on the walls of the Weyl chamber (see N. J. Weiss [**4**]).

We would like now to illustrate this result in the case of $SU(2)$ for which we use the notation of Coifman and Weiss [**3**]. The irreducible representations are indexed by the half integers $\lambda=0, \frac{1}{2}, 1, \frac{3}{2}, \cdots$, $d_\lambda=(2\lambda+1)$ and $\chi_\lambda(e(\theta))=\sin(2\lambda+1)\theta/\sin\theta$, where

$$e(\theta)=\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta}\end{pmatrix}.$$

In this case $\mathscr{D}m_\lambda=D_\alpha m_\lambda=m_\lambda-m_{\lambda-1}$. The theorem now reads as follows:

The sequence $\{m_\lambda\}$ is a multiplier for $L^p(SU(2))$ if the series

$$\sum\mathscr{D}[(2\lambda+1)m_\lambda]\cos(2\lambda+1)\theta$$

defines a bounded operator on L^p of the circle or, equivalently, $(2\lambda+1)m_\lambda-(2\lambda-1)m_{\lambda-1}=\mu_{2\lambda}$ is a multiplier for Fourier series.

As a consequence we obtain, by identifying special functions on $SU(2)$ as Jacobi polynomials $P_k^{\alpha,\beta}$, with α, β integers:

COROLLARY. *The operator M which assigns to the expansion*

$$f(x)=\sum_0^\infty \alpha_k P_k^{(\alpha,\beta)}(x)(1-x)^{\alpha/2}(1+x)^{\beta/2}$$

the expansion

$$M(f)(x)=\sum_0^\infty m_k\alpha_k P_k^{(\alpha,\beta)}(x)(1-x)^{\alpha/2}(1+x)^{\beta/2}$$

is bounded on $L^p([-1,1],dx)$ if the sequence $(k+1)m_k-(k-1)m_{k-1}$ defines an even L^p multiplier for Fourier series.

Another easy consequence is a theorem of Bonami and Clerc [**1**] stating that $\{m_\lambda\}$ is a multiplier on $SU(2)$ if $\sum_{2^N}^{2^{N+1}}\lambda|m_{\lambda+1}-2m_\lambda+m_{\lambda-1}|\leqq C$. This simply follows from our result and the theorem of Marcinkiewicz on the circle.

The proof of our theorem involves two steps. First we pass from a multiplier on G to a multiplier on T^l by means of an identity involving the Weyl character formula. Then the desired L^p inequalities are obtained by transferring inequalities on T^l to G as is done in [**2**]. A similar method is valid for symmetric spaces (not necessarily of compact type) reducing the study of spherical convolution operators to the study of associated operators on the group A appearing in an Iwasawa decomposition $G=KAN$. This will be done in a forthcoming paper.

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