BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 80, Number 1, January 1974

PERIODICALLY PERTURBED CONSERVATIVE SYSTEMS

BY SHAIR AHMAD

Communicated by Francois Treves, June 28, 1973

In this note we announce a result concerning the existence of a periodic solution for a class of periodically perturbed conservative systems. Our result, in a sense, completes a series of investigations originated by W. S. Loud [4]. Also see [1], [2], [3], and [5]. Our techniques are different from those of the authors cited above.

Consider the vector differential equation

$$x'' + \operatorname{grad} G(x) = p(t) = p(t + 2\pi), \tag{1}$$

where $p \in C(R, R^n)$, $G \in C^2(R^n, R)$. This equation can be interpreted as the newtonian equation of a mechanical system subject to conservative internal forces and periodical external forces.

THEOREM 1 (LAZER [1]). *Let A and B be real constant symmetric matrices such that if $\lambda_1 \leqq \lambda_2 \leqq \cdots \leqq \lambda_n$ and $\mu_1 \leqq \mu_2 \leqq \cdots \leqq \mu_n$ denote the eigenvalues of A and B respectively then there exist integers $N_k \geqq 0$, $k = 1, \cdots, n$, such that*

$$N_k^2 < \lambda_k \leqq \mu_k < (N_k + 1)^2.$$

If, for all $a \in R^n$, $A \leqq \partial^2 G(a)/\partial x_i\, \partial x_j \leqq B$, then (1) *has at most one 2π-periodic solution.*

Our theorem establishes the existence part of the preceding theorem. More specifically, we prove

THEOREM (1)*. *If G, A and B satisfy the hypothesis of Theorem* 1, *then* (1) *has a 2π-periodic solution.*

The key to the proof of our theorem is

LEMMA 1. *Let $\bar{Q}(t)$ be a real $n \times n$ symmetric matrix whose elements are bounded, measurable and 2π-periodic on the real line. Let A and B be real constant symmetric matrices such that $A \leqq \bar{Q}(t) \leqq B$. If $\lambda_1 \leqq \cdots \leqq \lambda_n$ and $\mu_1 \leqq \cdots \leqq \mu_n$ denote the eigenvalues of A and B respectively then there*

AMS (MOS) subject classifications (1970). Primary 34C25, 34A10.

exist integers $N_k \geqq 0$, $k=1, \cdots, n$, *satisfying*

$$N_k^2 < \lambda_k \leqq \mu_k < (N_k + 1)^2.$$

Let $f(t)$ *be a real vector-valued* 2π*-periodic continuous function with* $\|f(t)\| \leqq K$, K *some number. Then there exists a number* $r>0$, *independent of* $f(t)$, *such that for any periodic solution* u *of* $u''+\bar{Q}u=f$ *the inequality* $\|u(t)\|^2+\|u'(t)\|^2 \leqq r^2$ *holds for all* t (*we mean* u' *absolutely continuous and the preceding equation holds a.e.*).

Using this lemma we prove that our theorem follows from a generalization of Poincaré's perturbation theorem (see [3]). The proof of Lemma 1 is too long to give here. A brief sketch may be given along the following line. Assuming that the conclusion of Lemma 1 is false, we construct a sequence of equations of the form

$$z_m'' + Q_m(t)z_m = g_m(t) \quad \text{a.e.}$$

where z_m, Q_m and g_m are 2π-periodic (Q_m symmetric). It is shown that the sequences $\{z_m\}$ and $\{z_m'\}$ are uniformly bounded and equicontinuous, and $\{Q_m\}$ weakly converges to some matrix $Q(t)$. Using the fact that the set of symmetric $n\times n$ matrices S satisfying $A \leqq S \leqq B$ can be considered as a compact convex subset of R^p, $p=n(n+1)/2$, it follows from Lemma 1A of (p. 157 of [5]) that $Q(t)$ is a 2π-periodic symmetric matrix and $A \leqq Q(t) \leqq B$. It is then shown that this leads to a contradiction of Theorem 1 of [1].

References

1. A. C. Lazer, *Application of a lemma on bilinear forms to a problem in nonlinear oscillations*, Proc. Amer. Math. Soc. **33** (1972), 89–94. MR **45** #2258.

2. A. C. Lazer and D. A. Sánchez, *On periodically perturbed conservative systems*, Michigan Math. J. **16** (1969), 193–200. MR **39** #7212.

3. D. E. Leach, *On Poincarés perturbation theorem and a theorem of W. S. Loud*, J. Differential Equations **7** (1970), 34–53. MR **40** #4539.

4. W. S. Loud, *Periodic solutions of nonlinear differential equations of Duffing type*, Proc. United States–Japan Seminar on Differential and Functional Equations (Minneapolis, Minn., 1967), Benjamin, New York, 1967, pp. 199–224. MR **36** #6704.

5. E. B. Lee and L. Markus, *Foundations of optimal control theory*, Wiley, New York, 1967. MR **36** #3596.

Department of Mathematics, Oklahoma State University, Stillwater, Oklahoma 74074