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COMPLETION AND EMBEDDING BETWEEN PSEUDO (v, k, λ)-DESIGNS AND (v, k, λ)-DESIGNS

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ABSTRACT. Each of four arithmetical conditions on the parameters v, k, and λ of a given primary pseudo (v, k, λ)-design is necessary and sufficient to ensure completion or embedding between the given design and some (v', k', λ')-design.

Let $X=\{x_1, \cdots, x_v\}$, and let $X_1, \cdots, X_v$ be subsets of X. The subsets $X_1, \cdots, X_v$ are said to form a (v, k, λ)-*design* if

each X_j $(1 \leqq j \leqq v)$ has k elements;
any two distinct X_i, X_j $(1 \leqq i, j \leqq v)$ intersect in λ elements; and
$0 \leqq \lambda < k < v-1$.

Such a design is completely determined by its *incidence matrix*; this is the $(0, 1)$-matrix $A=[a_{ij}]$ defined by taking $a_{ij}=1$ if $x_j \in X_i$ and $a_{ij}=0$ if $x_j \notin X_i$. More information about these combinatorial designs is available, for example, in [**2**] and [**5**].

Let $Y=\{y_1, \cdots, y_v\}$, and let $Y_1, \cdots, Y_{v-1}$ be subsets of Y. The subsets $Y_1, \cdots, Y_{v-1}$ are said to form a *pseudo* (v, k, λ)-*design* if

each Y_j $(1 \leqq j \leqq v-1)$ has k elements;
any two distinct Y_i, Y_j $(1 \leqq i, j \leqq v-1)$ intersect in λ elements; and
$0 < \lambda < k < v-1$.

The incidence matrix of a pseudo (v, k, λ)-design is defined in the same manner as the incidence matrix of a (v, k, λ)-design.

The consideration of pseudo (v, k, λ)-designs was suggested during the course of study of "modular hadamard matrices" [**3**], [**4**]. Related work has been published by Bridges [**1**] and Woodall [**6**].

A pseudo (v, k, λ)-design is "almost" (its incidence matrix lacks one row) a (v, k, λ)-design; this suggests the consideration of "completion and embedding" between these two combinatorial designs. Let A be the incidence matrix of a pseudo (v, k, λ)-design. Then it *might* be possible to

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"complete" the $v-1$ rows of A by adjoining one additional row to A, and possibly performing some operations on the rows or columns of A, so that the incidence matrix of some (v, k', λ')-design is obtained; also, it *might* be possible that the incidence matrix of some $(v-1, k', \lambda')$-design is "embedded" in A. This paper presents a theorem and a conjecture dealing with such completion and embedding. No proof of the theorem below is given in this paper. A more comprehensive paper dealing with pseudo (v, k, λ)-designs is being planned by this author.

When $k=2\lambda$, the existence of a pseudo (v, k, λ)-design implies and is implied by the existence of some (v', k', λ')-design; and, if the parameters of a given pseudo (v, k, λ)-design satisfy $v\lambda=k^2$, then they must satisfy $k=2\lambda$ [3]. A pseudo (v, k, λ)-design is called *primary* or *nonprimary* according to whether its parameters satisfy $v\lambda\neq k^2$ or $v\lambda=k^2$, respectively. Thus, it is the existence of primary pseudo (v, k, λ)-designs which remains unresolved.

The incidence matrix of a pseudo (v, k, λ)-design can be obtained from the incidence matrix A of a given (v', k', λ')-design by any one of the following four simple techniques:

1. a column of $+1$'s is adjoined to A;
2. a column of 0's is adjoined to A;
3. a row is discarded from A; or
4. a row is discarded from A and then the k' columns of A which had a $+1$ in the discarded row are complemented (0's and $+1$'s are interchanged in these columns).

These four are the only known techniques for the construction of pseudo (v, k, λ)-designs. The initial observation which led to the theorem below is that there is a simple arithmetical condition on the parameters v, k, and λ which is *necessary* for the incidence matrix of a given primary pseudo (v, k, λ)-design to be obtained from the incidence matrix of some (v', k', λ')-design by one of the aforementioned techniques; it can be shown that each one of these conditions is *also sufficient*, thus answering the completion and embedding problem under consideration in these four cases.

THEOREM. *The incidence matrix of a given primary pseudo (v, k, λ)-design can be obtained from the incidence matrix of some (v', k', λ')-design by the ith $(1\leqq i\leqq 4)$ technique above if and only if the parameters v, k, and λ satisfy the respective ith condition below*:

1. $(k-1)(k-2)=(\lambda-1)(v-2)$;
2. $k(k-1)=\lambda(v-2)$;
3. $k(k-1)=\lambda(v-1)$; *or*
4. $k=2\lambda$.

A primary pseudo (v, k, λ)-design is said to be of *type i* $(1 \leqq i \leqq 4)$ if its parameters satisfy the ith equation in the statement of the above theorem. There are examples of pseudo (v, k, λ)-designs of each of these four types that are not of any of the other three types. It is possible for a pseudo (v, k, λ)-design to be of more than one type.

The condition that the parameters v, k, and λ satisfy the ith $(1 \leqq i \leqq 4)$ equation in the statement of the theorem above is *not* sufficient to ensure the existence of a pseudo (v, k, λ)-design, since none of these conditions is sufficient to ensure the existence of a (v', k', λ')-design with the appropriate parameters v', k', and λ'.

This author has conjectured that given a primary pseudo (v, k, λ)-design, then completion or embedding between the given design and some (v', k', λ')-design must always be possible. The precise statement is:

CONJECTURE. *The parameters of a given primary pseudo (v, k, λ)-design must satisfy at least one of the equations in the statement of the above theorem.*

It is known that the above conjecture is valid whenever $\lambda = 1$.

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