BEST APPROXIMATION BY COMPACT OPERATORS. II

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Introduction. Let H be an infinite-dimensional complex Hilbert space, and let $\mathscr{B}(H)$ (resp. $\mathscr{C}(H)$) be the algebra of all bounded (resp. compact) linear operators on H. It is well known [4], [6] that $\mathscr{C}(H)$ is proximinal in $\mathscr{B}(H)$, that is, every $T \in \mathscr{B}(H)$ has a best approximation from the subspace $\mathscr{C}(H)$. Indeed, it has recently been noted by Fakhoury [2] that there is a continuous selection for the associated metric projector. We denote the metric complement of $\mathscr{C}(H)$ in $\mathscr{B}(H)$ by $\mathscr{C}(H)^0$; by definition, $\mathscr{C}(H)^0 = \{T \in \mathscr{B}(H): \|T\| = \text{dist}(T, \mathscr{C}(H))\} \equiv \{T \in \mathscr{B}(H): 0 \text{ is a best compact approximation to } T\}$. Such operators (other than 0) are at maximum possible distance from $\mathscr{C}(H)$; they may thus be called "anti-compact" (called extremally noncompact by Coburn [1]). Every operator T is the sum of a compact operator and an anti-compact operator; we can give one such splitting explicitly in terms of the polar decomposition of T. It is the purpose of this note to announce a series of results on the contents and structure of $\mathscr{C}(H)^0$, with particular reference to the proper subset consisting of those operators for which 0 is the unique best compact approximation that commutes with the given operator. Full details and proofs of the theorems will appear elsewhere.

1. **Failure of unicity.** The seminal result that motivates and directs our subsequent investigation is the rather striking failure, in all cases, of uniqueness of best compact approximations.

THEOREM 1. *Let T be any noncompact operator in $\mathscr{B}(H)$. Then the set of all best compact approximations to T is a (closed, bounded) convex set of infinite dimension.*

It is clear that the metric complement of a Chebyshev subspace of a Banach space X is a (closed) nowhere dense subset of X, and simple examples show that the metric complement of a non-Chebyshev subspace

may well have nonempty interior. Thus the next result is unexpected (however, see Theorem 7 below):

THEOREM 2. $\mathscr{C}(H)^0$ is nowhere dense in $\mathscr{B}(H)$.

COROLLARY 1. The set of operators in $\mathscr{B}(H)$ which fail to attain their norm on the unit ball of H is nowhere dense in $\mathscr{B}(H)$.

The corollary derives its interest by contrast with a result of Lindenstrauss [5], which asserts that the set of operators which do attain their norm is dense in $\mathscr{B}(H)$ (this latter fact being valid if H is replaced by an arbitrary reflexive Banach space). Now from the corollary we see that the set of operators attaining their norm is actually of second category in $\mathscr{B}(H)$, and thus we obtain an independent proof of a slightly stronger form of Lindenstrauss' result in the Hilbert space case.

2. **Distance formulas.** In [3], [4] formulas were given for dist(T, $\mathscr{C}(H)$), for any $T \in \mathscr{B}(H)$. In particular, Gohberg and Kreĭn showed that dist(T, $\mathscr{C}(H)$) is the essential spectral radius of the positive part of T. We now consider a class of operators for which this result can be improved. Let $r_e(T)$ be the essential spectral radius of $T \in \mathscr{B}(H)$, and let us say that T is essentially normaloid if $r_e(T) = \|T\|$.

THEOREM 3. For every $T \in \mathscr{B}(H)$, we have $r_e(T) \leq \mathrm{dist}(T, \mathscr{C}(H))$. The class $\mathscr{D}$ of operators T for which equality holds here is a closed, balanced, nowhere dense cone in $\mathscr{B}(H)$. $\mathscr{D}$ consists of those operators having an essentially normaloid compact perturbation, and in particular, $\mathscr{D}$ contains every seminormal operator.

Since this result is established independently of [3], and since dist(T,$\mathscr{C}(H)$)=dist((T^*T)$^{1/2}$, $\mathscr{C}(H)$) always holds, it follows that Theorem 3 includes the Gohberg-Kreĭn formula, and hence Theorem 5.1 of [1]. That $\mathscr{D}$ contains every seminormal operator is an easy consequence of the work of Salinas [7]. On the other hand, $\mathscr{D}$ does not contain all weighted shifts.

THEOREM 4. Let T be a (unilateral or bilateral) weighted shift with weights $a_n > 0$. Then

$$\mathrm{dist}(T, \mathscr{C}(H)) = \limsup_{|n| \to \infty} a_n.$$

3. **Anti-compact operators.** What kinds of operators in $\mathscr{B}(H)$ are anti-compact? From [4] it follows, for example, that all partial isometries of infinite rank are anti-compact. We also have

THEOREM 5. Every essentially normaloid operator is anti-compact.

These two types of operators do not yet exhaust the metric complement $\mathscr{C}(H)^0$. For example, certain quasi-nilpotent weighted shifts are anti-compact.

THEOREM 6.   *The metric complement $\mathscr{C}(H)^0$ contains a homeomorph of the Calkin algebra.*

It is an open question whether $\mathscr{C}(H)^0$ is actually homeomorphic to a Banach space.

THEOREM 7. $\mathscr{C}(H)^0$ *is strong operator dense in $\mathscr{B}(H)$.*

4. Uniqueness of commuting best approximations.   The proof of Theorem 3 shows that every seminormal operator T has a best compact approximation that commutes with T. Motivated by this observation and by Theorem 1 we ask which operators have unique *commuting* best compact approximations. Now in general, on account of Lomonosov's theorem [6], this is not a trivial problem. However, anti-compact operators have, by definition, a commuting best compact approximation, namely 0. To study this situation in more detail, we introduce two classes of operators in $\mathscr{B}(H)$: $ZUC=\{T \in \mathscr{B}(H):0$ is the unique compact operator that commutes with $T\}$ and $ZUCA=\{T \in \mathscr{B}(H):0$ is the unique commuting best compact approximation to $T\}$. Clearly,

$$ZUC \cap \mathscr{C}(H)^0 \subset ZUCA \subset \mathscr{C}(H)^0,$$

and these inclusions are proper.

Below we identify several interesting classes of $ZUCA$ operators. These operators are all subnormal and eigenvalueless, hence are at least anti-compact by Theorem 5. Further examples of $ZUCA$ operators are obtained as the adjoints of these operators, and hence may be only seminormal and have nonempty point spectrum. These examples should be contrasted with

THEOREM 8.   *No hyponormal operator with nonempty point spectrum can have a unique commuting best compact approximation.*

THEOREM 9.   *Each of the following classes of operators belongs to $ZUC \cap \mathscr{C}(H)^0$:*
  (a) *normal and eigenvalueless;*
  (b) *isometric and eigenvalueless;*
  (c) *analytic Toeplitz operators, defined by a schlicht function;*
  (d) *multiplications by bounded schlicht functions on some Bergman space;*
  (e) *the discrete Cesàro operator.*

The following general principle is useful for some of these cases.

LEMMA. *Let T be an eigenvalueless operator in $\mathscr{B}(H)$ such that T^* has so many simple eigenvalues that the corresponding eigenvectors are fundamental in H. Then T belongs to ZUC (although not necessarily to $\mathscr{C}(H)^0$).*

Note that, although the result of Shields and Wallen [**8**, Theorem 2] implies that their multiplication operators M_z belong to ZUC, the preceding lemma is applicable to a more general situation where their condition (c) is significantly weakened and condition (d) is eliminated.

Unfortunately, it is not the case that every eigenvalueless subnormal operator belongs to $ZUCA$. In fact, we have

EXAMPLE.   There exists an eigenvalueless quasi-normal operator not in the class $ZUCA$.

To see that the class $ZUCA$ properly contains the class $ZUC \cap \mathscr{C}(H)^0$, let U be the ordinary unilateral shift, and consider the operator $2U \oplus U$. More complicated examples exist: these are certain eigenvalueless weighted shifts that fail to attain their norm (compare with Corollary 3 below). In general, any weighted shift in either ZUC or $ZUCA$ must be eigenvalueless.

THEOREM 10. *A shift with weights $a_n > 0$ fails to belong to ZUC exactly when there exists some $k \geq 1$ such that*

$$\lim_{n \to \infty} (a_{n+1} a_{n+2} \cdots a_{n+k}) = 0.$$

COROLLARY 2. *Every hyponormal weighted shift belongs to $ZUC \cap \mathscr{C}(H)^0$.*

Also, as a particular consequence of a (rather complicated) sufficient condition for a weighted shift to belong to $ZUCA$, we have

COROLLARY 3. *Every anti-compact weighted shift that attains its norm belongs to $ZUCA$.*

The relationship between the metric complement $\mathscr{C}(H)^0$ and its subset $ZUCA$ is interesting. For example, if a nonzero operator belongs to $ZUCA$, it must not admit any compact direct summand; no stronger necessary condition is known. In a different vein, the possibility that $ZUCA$ is dense in $\mathscr{C}(H)^0$ is an intriguing but open question. Certainly $ZUCA$ is not closed. There is, for example, a sequence of eigenvalueless normal operators that converges to the identity operator. Such a phenomenon exhibits the delicate and discontinuous nature of the $ZUCA$ property: we have a sequence of operators, each of whose set of commuting best compact approximations is zero-dimensional, but whose (norm) limit has an infinite-dimensional set of commuting best compact approximations.

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