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STABILITY AND TRANSVERSALITY

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1. **Introduction.** Let N and P be C^∞ manifolds of dimensions n and p and let $C^\infty(N, P)$ denote the space of all C^∞ mappings $f: N \to P$ with the fine C^∞ topology [**2**, II, p. 259]. A mapping $f \in C^\infty(N, P)$ may be stable in either the C^∞ [**2**, II] or topological [**3**] sense. In this paper we state certain results connecting these two concepts of stability. In a related development we also outline a procedure for showing that topologically stable mappings satisfy certain transversality conditions. All of the results given here are based on our thesis [**4**] to which we refer for proofs and further details.

2. **A conjecture.** It is clear that any C^∞ stable mapping is also topologically stable, but the converse is false in general. In fact for N compact Mather has shown that the topologically stable mappings are always dense in $C^\infty(N, P)$ [**3**], while the C^∞ stable mappings are dense if and only if n, p lie in a certain "nice" range [**2**, VI]. However, one may still conjecture the following:

(2.1) *If N is compact and n, p lie in the "nice" range, then any topologically stable mapping*

$$f: N \to P$$

is also C^∞ stable.

In [**4**] we verify the above conjecture for the comparatively simple cases $p > 2n$ ("Whitney embedding" range) and $p = 1$ ("functions"). We obtain related results for a more substantial range of dimensions by introducing a "uniform stability" condition.

DEFINITION. $f \in C^\infty(N, P)$ is *uniformly stable* provided that for any family

$$F: (\boldsymbol{R}^K, 0) \to (C^\infty(N, P), f)$$

of maps (parameterized by $\boldsymbol{R}^K$, any $K > 0$) for which the associated map

$$\tilde{F}: N \times \boldsymbol{R}^K \to P \times \boldsymbol{R}^K$$

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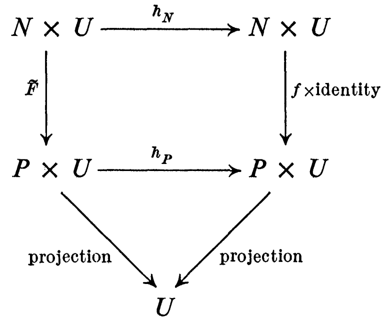
is C^∞, there exists a neighborhood U of $0 \in \boldsymbol{R}^K$ and homeomorphisms h_N, h_P for which the following diagram commutes:



A major result of [**4**] can now be stated as follows:

THEOREM 2.2. *Let N be compact and assume $n>p$, $p<7$, $n<2(n-p+2)$. Let $f \in C^\infty(N, P)$ be*

(a) *topologically stable, and*

(b) *in the interior of the set of uniformly stable maps.*

Then f is C^∞ stable.

3. **A more general problem.** In view of Mather's characterization [**2**, V] of C^∞ stable mappings, the major task in proving Theorem 2.2 is to show that f is transverse to each orbit in $J^{p+1}(N, P)$ of the group $\mathscr{K}^{p+1}$ of [**2**, IV]. We are then led to a more general question treated in [**4**]. Let $J^k(n, p)$ be the space of k-jets at 0 of C^∞ mappings $f:(\boldsymbol{R}^n, 0) \to (\boldsymbol{R}^p, 0)$, and let Σ be a submanifold of $J^k(n, p)$ which is invariant under the group $\mathscr{K}^k$. Then for any N, P we have an associated subbundle $\Sigma(N, P) \subseteq J^k(N, P)$ with fiber Σ. We would like to know whether for any compact N and topologically stable $f: N \to P$ we have $j^k f$ transverse to $\Sigma(N, P)$. In the next sections we outline the procedure of [**4**] for attacking this problem.

4. **Topological transversality.** We first replace transversality to $\Sigma(N, P)$ by a more "topological" concept. Given $\Sigma \subseteq J^k(N, P)$ and $f \in C^\infty(N, P)$, define $\Sigma(f) \subseteq N$ by $\Sigma(f) = (j^k f)^{-1}[\Sigma]$. Also for any submanifold $\Sigma \subseteq J^k(N, P)$ define

$$\operatorname{cod} \Sigma = \text{dimension } J^k(N, P) - \text{dimension } \Sigma.$$

DEFINITION. Let Σ be a submanifold of $J^k(N, P)$ and let $f \in C^\infty(N, P)$. Then f is *topologically transverse* to Σ at $x \in N$ if either $x \notin \Sigma(f)$, or

Case A. $n > \operatorname{cod} \Sigma$ and there exist neighborhoods U of x, W of f, such that $\Sigma(g) \cap U$ is a topological manifold of dimension $n-\text{cod}(\Sigma)$ for all $g \in W$, or

Case B. $n=\text{cod}(\Sigma)$ and there exist neighborhoods U of x, W of f, such that $\Sigma(g) \cap U$ is a single point for all $g \in W$.

It follows from familiar properties of transversal maps that transversality$\Rightarrow$topological transversality for any f and Σ. The converse is unclear, but we have proved the following [**4**]:

PROPOSITION 4.1. *Let Σ be a $\mathscr{K}^k$-invariant submanifold of $J^k(n, p)$ which is contained in a Boardman singularity* [**1**] *of the form*

$$\Sigma^{i_1, i_2, \ldots, i_k}, \qquad i_k = 0.$$

Then a map $f: N \to P$ is transverse to $\Sigma(N, P)$ if and only if f is topologically transverse to $\Sigma(N, P)$.

PROPOSITION 4.2. *Let $\Sigma^i \subseteq J^1(n, p)$ be a first order Boardman singularity* [**1**]. *Then a map $f: N \to P$ is transverse to $\Sigma^i(N, P)$ if and only if f is topologically transverse to $\Sigma^i(N, P)$.*

5. **Germ classes.** Let $C^\infty(n, p)$ denote the set of germs $[f]$ at 0 of C^∞ mappings $f: (\boldsymbol{R}^n, 0) \to (\boldsymbol{R}^p, 0)$. $[f]$, $[g] \in C^\infty(n, p)$ are C^∞ (respectively, topologically) equivalent if there exist diffeomorphisms (respectively, homeomorphisms) $h_n: (\boldsymbol{R}^n, 0) \to (\boldsymbol{R}^n, 0)$, $h_p: (\boldsymbol{R}^p, 0) \to (\boldsymbol{R}^p, 0)$ such that $[f] = [h_p^{-1} \circ g \circ h_n]$. A subset $\Sigma \subseteq C^\infty(n,p)$ is a C^∞ (respectively, topological) germ class if $[f] \in \Sigma \Rightarrow [g] \in \Sigma$ for any $[g]$ which is C^∞ (respectively, topologically) equivalent to $[f]$. If Σ is a C^∞ (or topological) germ class and $f \in C^\infty(N, P)$, we can define $\Sigma(f) \subseteq N$ by

$$x \in \Sigma(f) \Leftrightarrow [\varphi \circ f \circ \psi^{-1}] \in \Sigma$$

where $\psi: (U, x) \to (\boldsymbol{R}^n, 0)$ and $\varphi: (V, f(x)) \to (\boldsymbol{R}^p, 0)$ are local coordinates on N, P.

We call a C^∞ germ class $\Lambda \subseteq C^\infty(n, p)$ *generic* if $\{f \in C^\infty(N, P) | \Lambda(f) = N\}$ is dense in $C^\infty(N, P)$ for any N, P. For example the set of germs of all maps satisfying a countable number of transversality conditions will be generic by the Thom transversality theorem [**2**, V].

In [**4**] we prove the following basic results relating topological transversality to the existence of appropriate topological germ classes. For these results we assume the source manifold N is *compact*.

PROPOSITION 5.1. *Let Σ be a $\mathscr{K}^k$-invariant submanifold of $J^k(n, p)$ and Σ_{Top} a topological germ class in $C^\infty(n, p)$ such that $\Sigma(f) = \Sigma_{\text{Top}}(f)$ for any topologically stable mapping $f \in C^\infty(N, P)$. Then any topologically stable* *map* $f \in C^\infty(N, P)$ *is topologically transverse to* $\Sigma(N, P)$ *at every point* $x \in N$.

PROPOSITION 5.2. *Let* $\Sigma \subseteq C^\infty(n, p)$ *be a closed* C^∞ *germ class and* Σ_{Top} *a topological germ class. Assume there exists a generic class* $\Lambda \subseteq C^\infty(n, p)$ *such that*

(i) $\Lambda \cap \Sigma_{\mathrm{Top}} = \Lambda \cap \Sigma$;

(ii) $\Lambda \cap \Sigma$ *is dense in* Σ;

(iii) *for any* $[g] \in \Lambda$ *and any open* U *containing* $0 \in \mathbf{R}^n$, *there exists an open* $U' \subseteq U$, $0 \in U'$, *such that* $U' \cap \Sigma(g)$ *is connected.*

Then for any N, P *and topologically stable* $f \in C^\infty(N, P)$, *we have* $\Sigma(f) = \Sigma_{\mathrm{Top}}(f)$.

6. **Summary.** Our procedure for proving transversality properties of topologically stable mappings is then as follows: Given a $\mathscr{K}^k$-invariant submanifold $\Sigma \subseteq J^k(n, p)$, we write $\Sigma = \Sigma_1 - \Sigma_2$, where Σ_1, Σ_2 denote closed, $\mathscr{K}^k$-invariant subsets of $J^k(n, p)$ and also the C^∞ germ classes in $C^\infty(n, p)$ corresponding to Σ_1, Σ_2. We next find topological germ classes $\Sigma_{1,\mathrm{Top}}$, $\Sigma_{2,\mathrm{Top}}$ such that the hypotheses of Proposition 5.2 are satisfied by Σ_1, $\Sigma_{1,\mathrm{Top}}$ and Σ_2, $\Sigma_{2,\mathrm{Top}}$. It follows that

$$\Sigma(f) = \Sigma_1(f) - \Sigma_2(f) = (\Sigma_{1,\mathrm{Top}} - \Sigma_{2,\mathrm{Top}})(f)$$

for any topologically stable f. But then by Proposition 5.1 any topologically stable f is topologically transverse to Σ. Finally, if Σ satisfies the conditions of Proposition 4.1 or 4.2 we have f transverse to Σ for any topologically stable f.

In [**4**] the above program is carried out for various Σ. For example we show

PROPOSITION 6.1. *Let* $\Sigma^i \subseteq J^1(n, p)$ *be a first order Boardman singularity with* $n \geqq \operatorname{cod} \Sigma^i$. *Then for any* N, P, N *compact, and any topologically stable* $f \in C^\infty(N, P)$, *we have* f *transverse to* $\Sigma^i(N, P)$.

Also, for the range of dimensions considered in Theorem 2.2 we use the above technique to show that any topologically stable $f \in C^\infty(N, P)$ is transverse to $\Sigma(N, P)$ for any $\mathscr{K}^{p+1}$-orbit $\Sigma \subseteq J^{p+1}(n, p)$, provided N is compact and $n \geqq \operatorname{cod} \Sigma$. (The uniform stability condition (b) of Theorem 2.2 is then used only to show that $j^{p+1}f \cap \Sigma = \varnothing$ for those Σ with $n < \operatorname{cod} \Sigma$.)

REMARK. When N is not compact, a simple counterexample given in [**4**] shows that Theorem 2.2 (and Proposition 6.1) fail to hold even for proper mappings. However, analogous results are obtained in [**4**] for the noncompact case by replacing the condition of topological stability by that of ε-stability.

BIBLIOGRAPHY

1. J. M. Boardman, *Singularities of differentiable mappings*, Inst. Hautes Études Sci. Publ. Math. **33** (1967), 21–57. MR **37** #6945.

2. J. N. Mather, *Stability of C^∞ mappings*. II, IV, V, VI, Ann. of Math. (2) **89** (1969), 254–291; Inst. Hautes Études Sci. Publ. Math. No. **37** (1969), 223–248; Advances in Math. **4** (1970), 301–336; Proceedings of the Liverpool Singularities-Symposium, 1 (1969/70), Lecture Notes in Math., vol. 192, Springer, Berlin, 1971, pp. 207–253. MR **41** #4582; **43** #1215b, c; **45** #2747.

3. ———, *Stratifications and mappings*, Proceedings of Dynamical Systems, Salvadore, Brazil, July 1971 (to appear).

4. R. D. May, *Transversality properties of topologically stable mappings*, Ph.D. thesis, Harvard University, Cambridge, Mass., 1973.

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