BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 80, Number 1, January 1974

A STRUCTURE THEORY OF JORDAN PAIRS

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Communicated by Nathan Jacobson, July 6, 1973

1. **Definitions.** Let V^σ be modules over a ring k where $\sigma=\pm$. For a quadratic map $Q^\sigma: V^\sigma\to\mathrm{Hom}_k(V^{-\sigma}, V^\sigma)$ let

$$L^\sigma(x, y)z = Q^\sigma(x+z)y - Q^\sigma(x)y - Q^\sigma(z)y = \{xyz\}.$$

A *Jordan pair* over k is a pair $\mathscr{V}=(V^+, V^-)$ of k-modules together with a pair (Q^+, Q^-) of quadratic maps $Q^\sigma: V^\sigma\to\mathrm{Hom}_k(V^{-\sigma}, V^\sigma)$ such that the identities

$$\text{(JP1)} \qquad L^\sigma(x, y)Q^\sigma(x) = Q^\sigma(x)L^{-\sigma}(y, x),$$

$$\text{(JP2)} \qquad L^\sigma(Q^\sigma(x)y, y) = L^\sigma(x, Q^{-\sigma}(y)x),$$

$$\text{(JP3)} \qquad Q^\sigma(Q^\sigma(x)y) = Q^\sigma(x)Q^{-\sigma}(y)Q^\sigma(x),$$

hold in all base ring extensions. Jordan pairs have first been studied by K. Meyberg in [**6**], although not in the present form.

A *homomorphism* $h:\mathscr{V}\to\mathscr{W}$ of Jordan pairs is a pair $h=(h^+, h^-)$ of k-linear maps $h^\sigma: V^\sigma\to W^\sigma$ such that $h^\sigma Q^\sigma(x)=Q^\sigma(h^\sigma(x))h^{-\sigma}$, for all $x\in V^\sigma$. The *opposite* of $\mathscr{V}$ is $\mathscr{V}^{\mathrm{op}}=(V^-, V^+)$ with quadratic maps (Q^-, Q^+). An *antihomomorphism* from $\mathscr{V}$ to $\mathscr{W}$ is a homomorphism from $\mathscr{V}$ to $\mathscr{W}^{\mathrm{op}}$. An antiautomorphism η of $\mathscr{V}$ is called an *involution* if $\eta^{-\sigma}\eta^\sigma$ is the identity on V^σ.

2. **Connections with Jordan algebras and Jordan triple systems.** There is a one-to-one correspondence between Jordan triple systems (cf. [**7**]) and Jordan pairs with involution as follows: If η is an involution of the Jordan pair $\mathscr{V}$ then V^+ becomes a Jordan triple system with quadratic operators $P(x)=Q^+(x)\eta^+$. If conversely (V, P) is a Jordan triple system then (V, V) is a Jordan pair with $Q^\sigma(x)y=P(x)y$ and involution $\eta^\sigma=\mathrm{Id}_V$. The structure group of the Jordan triple system is the automorphism group of the corresponding Jordan pair.

Let $\mathscr{V}$ be a Jordan pair. An element $a\in V^+$ is called invertible if $Q^+(a)$ is invertible. There is a one-to-one correspondence between Jordan pairs containing invertible elements and isotopism classes of unital quadratic

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AMS (MOS) subject classifications (1970). Primary 17C10, 17C20.

Key words and phrases. Jordan pair, Jordan algebra, alternative pair, inner ideals, chain condition, classification.

Jordan algebras: if $a \in V^+$ is invertible then $\mathscr{J}=(V^+, U^{(a)}, a)$ where $U^{(a)}(x)=Q^+(x)Q^+(a)^{-1}$ is a unital quadratic Jordan algebra. If b is also invertible then $U^{(b)}(x)=U^{(a)}(x)U^{(a)}(b)^{-1}$ and therefore $(V^+, U^{(b)}, b)$ is an isotope of $\mathscr{J}$. Conversely, if (J, U, e) is a unital quadratic Jordan algebra then (J, U) is a Jordan triple system, and the corresponding Jordan pair contains invertible elements. Again the structure group of the Jordan algebra is the automorphism group of the corresponding Jordan pair.

3. **The radical.** Let $\mathscr{V}$ be a Jordan pair and let $y \in V^{-\sigma}$. The k-module V^σ becomes a quadratic Jordan algebra, denoted V^σ_y, with quadratic operators $U(x)=Q^\sigma(x)Q^{-\sigma}(y)$ and squaring $x^2=Q^\sigma(x)y$. A pair $(x, y) \in V^\sigma \times V^{-\sigma}$ is called *quasi-invertible* if x is quasi-invertible in V^σ_y. The *Jacobson radical* of $\mathscr{V}$ is $\operatorname{Rad}\mathscr{V}=(\operatorname{Rad} V^+, \operatorname{Rad} V^-)$ where $\operatorname{Rad} V^\sigma=\{x \in \mathscr{V}^\sigma : (x, y) \text{ quasi-invertible for all } y \in V^{-\sigma}\}$. It has the same properties as the radical of a Jordan algebra (cf. [**2**]).

4. **Peirce decomposition.** An element $x \in V^\sigma$ is called (von Neumann-) *regular* if $x=Q^\sigma(x)y$ for some $y \in V^{-\sigma}$. An *idempotent* of $\mathscr{V}$ is a pair $c=(c^+, c^-) \in V^+ \times V^-$ such that $c^\sigma=Q^\sigma(c^\sigma)c^{-\sigma}$, i.e., c^σ is an idempotent in $V^\sigma_{c^{-\sigma}}$.

LEMMA. *If $c^+ \in V^+$ is regular then there exists $c^- \in V^-$ such that (c^+, c^-) is an idempotent.*

Indeed, if $c^+=Q^+(c^+)y$ set $c^-=Q^-(y)c^+$. Then it follows from (JP3) that (c^+, c^-) is an idempotent. For an idempotent c we have the *Peirce decomposition* $\mathscr{V}=\mathscr{V}_1(c)\oplus\mathscr{V}_{1/2}(c)\oplus\mathscr{V}_0(c)$ where $\mathscr{V}_i(c)=(V_i^+, V_i^-)$ is a subpair of $\mathscr{V}$ and V_i^σ is the Peirce-i-space of the Jordan algebra $V^\sigma_{c^{-\sigma}}$. Also V_1^σ is a unital quadratic Jordan algebra (as subalgebra of $V^\sigma_{c^{-\sigma}}$) with unit element c^σ. Two idempotents c and d are called *orthogonal* if $d \in \mathscr{V}_0(c)$. As for Jordan algebras there is a Peirce decomposition with respect to a system of pairwise orthogonal idempotents.

5. **Inner ideals.** A submodule I of V^σ is called an *inner ideal* if $Q^\sigma(I)V^{-\sigma} \subset I$. An element $x \in V^\sigma$ is called an *absolute zero divisor* if $Q^\sigma(x)=0$. Let $0 \neq I \subset V^+$ be a minimal inner ideal. Then either $I=k \cdot x$ where x is an absolute zero divisor, or $I=Q^+(x)V^-$ for all $0 \neq x \in I$; in particular, I consists of regular elements. In the latter case, let $0 \neq c^+ \in I$ and choose $c^- \in V^-$ such that (c^+, c^-) is an idempotent which is possible by the Lemma. Then $I=V_1^+$ is a Jordan division algebra. This shows that there are no minimal inner ideals of type II (cf. [**2**]) for Jordan pairs.

6. **Artinian Jordan pairs.** The *principal inner ideal* generated by an element $x \in V^\sigma$ is $Q^\sigma(x)V^{-\sigma}$. A Jordan pair $\mathscr{V}$ is called *Artinian* if V^σ satisfies the descending chain condition for principal inner ideals.

THEOREM 1. *Let $\mathscr{V}$ be Artinian. Then there exists an idempotent c such that $\mathscr{V}_0(c) \subset \operatorname{Rad} \mathscr{V}$ and $c = c_1 + \cdots + c_n$ is the sum of orthogonal primitive idempotents. If $\operatorname{Rad} \mathscr{V} = 0$ then c is maximal (i.e., $\mathscr{V}_0(c) = 0$) and the c_i are completely primitive.*

Here an idempotent c is called *primitive* if c^σ is the only idempotent in V_1^σ, and *completely primitive* if V_1^σ is a Jordan division algebra. Thus in contrast to Jordan algebras, a Jordan pair always has capacity.

For $\mathscr{V}$ Artinian the following conditions are equivalent:

(i) $\mathscr{V}$ is regular; i.e., V^σ consists of regular elements.

(ii) $\operatorname{Rad} \mathscr{V} = 0$.

(iii) V^σ contains no absolute zero divisors $\neq 0$.

THEOREM 2. *The regular Artinian Jordan pairs are precisely the finite direct products of simple regular Artinian Jordan pairs.*

Conversely we conjecture: A simple Artinian Jordan pair is regular. This has been proved in the finite-dimensional case.

7. **Alternative pairs.** A pair $\mathscr{A} = (A^+, A^-)$ of k-modules together with trilinear maps $A^\sigma \times A^{-\sigma} \times A^\sigma \to A^\sigma$, $(x, y, z) \mapsto \langle xyz \rangle$, is called an *alternative pair* over k if the identities

$$\text{(AP1)} \qquad \langle uv\langle xyz\rangle\rangle + \langle xy\langle uvz\rangle\rangle = \langle\langle uvx\rangle yz\rangle + \langle x\langle vuy\rangle z\rangle,$$

$$\text{(AP2)} \qquad \langle xy\langle xyz\rangle\rangle = \langle\langle xyx\rangle yz\rangle,$$

$$\text{(AP3)} \qquad \langle uv\langle xyx\rangle\rangle = \langle\langle uvx\rangle yx\rangle,$$

hold for all $x, u, z \in A^\sigma$, $y, v \in A^{-\sigma}$. We say $\mathscr{A}$ is *associative* if the identity

$$\langle uv\langle xyz\rangle\rangle = \langle u\langle yxv\rangle z\rangle = \langle\langle uvx\rangle yz\rangle$$

is satisfied. The relation between alternative pairs and Jordan pairs is as follows: If c is a maximal idempotent of the Jordan pair $\mathscr{V}$ then $\mathscr{V}_{1/2}(c) = (V_{1/2}^+, V_{1/2}^-)$ is an alternative pair with $\langle xyz \rangle = \{\{xyc^\sigma\}c^{-\sigma}z\}$. Conversely, every alternative pair $\mathscr{A}$ may be realized as the Peirce-$\frac{1}{2}$-space of a Jordan pair $\mathscr{V} = \mathscr{V}(\mathscr{A})$ with respect to a maximal idempotent. The construction of $\mathscr{V}(\mathscr{A})$ is similar to the one for alternative triple systems (cf. [**4**], [**7**]).

Every alternative pair $\mathscr{A}$ gives rise to a Jordan pair $\mathscr{A}^J$ having the same underlying modules and quadratic maps $Q^\sigma(x)y = \langle xyx \rangle$. The concepts of radical, idempotent, (principal) inner ideal, and Artinian are the same for $\mathscr{A}$ and $\mathscr{A}^J$.

8. **Classification.** Let $\mathscr{V}$ be a regular simple Artinian Jordan pair over k. Since these properties are invariant under restriction of the ring of scalars we may assume $k = \boldsymbol{Z}$. Let $\mathscr{V} = \mathscr{V}_1(c) \oplus \mathscr{V}_{1/2}(c)$ with respect to a maximal idempotent. If $\mathscr{V}_{1/2}(c) = 0$ then $\mathscr{V}$ contains invertible elements

and is therefore essentially a unital quadratic Jordan algebra. The structure of these algebras is well known ([1], [2], [5]). If $\mathscr{A}=\mathscr{V}_{1/2}(c)\neq 0$ then $\mathscr{A}$ is a simple regular Artinian alternative pair, and $\mathscr{V}=\mathscr{V}(\mathscr{A})$. For this reason we first classify alternative pairs.

THEOREM 3. *The simple regular Artinian alternative pairs are up to isomorphism precisely the following.*

(Ia) (M^+, M^-) *where* M^σ *is a left module over a simple associative Artinian ring* R^σ, *with product* $\langle xyz\rangle=\varphi^\sigma(x, y)z$. *Here* $R^{-\sigma}$ *is anti-isomorphic with* R^σ *under an anti-isomorphism* $a\mapsto\bar{a}$, *and* $\varphi^\sigma: M^\sigma\times M^{-\sigma}\to R^\sigma$ *is sesquilinear and nondegenerate, i.e., it is* R^σ*-linear in the first variable,* $\overline{\varphi^\sigma(x, y)}=\varphi^{-\sigma}(y, x)$, *and* $\varphi^\sigma(x, M^{-\sigma})=0$ *implies* $x=0$.

(Ib) (M^+, M^-) *as above but with product* $\langle xyz\rangle=\varphi^\sigma(z, y)x$.

(II) (C^+, C^-) *where* $C=C^+$ *is a Cayley algebra,* $C^-=C^{\mathrm{op}}$, *and* $\langle xyz\rangle=(x\bar{y})z$. *Here* $x\mapsto\bar{x}$ *denotes the canonical anti-isomorphism* $C^\sigma\to C^{-\sigma}$.

(III) (A^+, A^-) *where* A^σ *is a vector space over a field* K, *with product* $\langle xyz\rangle=\alpha^\sigma(z, I^\sigma(y))\cdot x+I^\sigma(y)\alpha^\sigma(x, z)$. *Here* α^σ *is a nondegenerate alternating bilinear form on* A^σ, *and* $I^\sigma: A^{-\sigma}\to A^\sigma$ *is a* K*-linear map leaving* α^σ *invariant and satisfying* $I^\sigma\circ I^{-\sigma}=-\mathrm{Id}$.

The pairs of type (I) *are associative, the others are properly alternative.*

From this we compute the associated Jordan pairs $\mathscr{V}=\mathscr{V}(\mathscr{A})$. For $\mathscr{A}$ of type (Ib) with M^σ not finitely generated as R^σ-module and of type (III) with A^σ infinite dimensional $\mathscr{V}(\mathscr{A})$ is not simple. Using the well-known classification of simple Artinian Jordan algebras we have the following result (where convenient, Jordan algebras or triple systems are listed instead of the corresponding Jordan pair).

THEOREM 4. *The simple regular Artinian Jordan pairs are up to isomorphism precisely the following.*

(0) *Jordan division algebras.*

(I) $\mathscr{A}^J$ *where* $\mathscr{A}$ *is an alternative pair of type* (Ia).

(II) *Alternating* $(n\times n)$*-matrices over a field,* $n\geqq 2$.

(III) *Outer ideals containing* 1 *in Jordan algebras of hermitian* $(n\times n)$*-matrices over an associative division algebra with involution,* $n\geqq 2$.

(IV) *Outer ideals containing* 1 *in Jordan algebras of nondegenerate quadratic forms over a field.*

(V) $(M_{1,2}(C), M_{1,2}(C^{\mathrm{op}}))$, (1×2)*-matrices over a Cayley algebra* C, *resp.* C^{op}, *with* $Q^\sigma(x)y=x({}^t\bar{y}x)$. *Here* ${}^t\bar{y}$ *denotes the transpose of* y *with coefficients in the opposite algebra.*

(VI) *Hermitian* (3×3)*-matrices over a Cayley algebra.*

Details will appear elsewhere.

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