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DYNAMICAL SYSTEMS, FILTRATIONS AND ENTROPY

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Introduction. I would like to try to pose some new directions in dynamical systems, at least in that part of the subject which deals with the qualitative behavior of the orbit structure of diffeomorphisms and flows of compact differentiable manifolds M without boundary. The birth of our first child, Alexander, to Beth and myself just two weeks before this talk has perhaps made me overly optimistic. But, I think that the subject can use some general approaches even if they be Pollyannic.

The basic problems that I will consider are:

(I) *Genericity*: That is, to find a generic set of diffeomorphisms or flows such that the asymptotic behavior of the orbits can somehow be reasonably understood.

A generic set means a subset which is a Baire set, that is, the countable intersection of open and dense sets.

(II) *Model making*: The idea here is to produce in some sense the best or simplest diffeomorphisms in each isotopy class of diffeomorphisms of M, that is, in each connected component of $\mathrm{Diff}^r(M)$, the group of C^r diffeomorphisms of M, $1 \leqq r \leqq \infty$. The properties that we shall say make a diffeomorphism f simplest are:

(a) f is structurally stable;

(b) f has the smallest topological entropy of any structurally stable diffeomorphism in its isotopy class.

Recall that $f \in \mathrm{Diff}^r(M)$ is structurally stable if there is a neighborhood of f, $U_f \subset \mathrm{Diff}^r(M)$, such that for any $g \in U_f$ there is a homeomorphism $h: M \to M$ with $hf = gh$. Structural stability says that up to continuous changes of variables the orbit structure of the diffeomorphisms in a neighborhood of f is locally constant.

Now to define entropy via a theorem of Bowen [**4**]. Let (X, d) be a compact metric space and $T: X \to X$ continuous. A set $E \subset X$ is (n, ε) separated if for any $x, y \in E$ with $x \neq y$ there is a j, $0 \leqq j < n$, such that $d(T^j(x), T^j(y)) > \varepsilon$. Let $S_n(\varepsilon)$ denote the largest cardinality of any (n, ε)

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separated set in X, and let

$$S_\varepsilon(T) = \limsup n^{-1} \log S_n(\varepsilon).$$

The topological entropy of T, $h(T)$ is then given by the formula:

$$h(T) = \lim_{\varepsilon \to 0} S_\varepsilon(T).$$

For $f \in \mathrm{Diff}^r(M)$, $0 \leq h(f) < \infty$ [**4**]. So the topological entropy essentially gives the asymptotic exponential growth rate of the number of orbits of f up to any accuracy and arbitrarily high period. In some sense then, the simplest diffeomorphisms are the ones which are structurally stable and which have the fewest orbits.

The work I shall be describing below was almost entirely done in collaboration with Z. Nitecki, S. Smale, and D. Sullivan and is contained in [**11**], [**18**], and [**19**]. Besides these mathematicians I have also benefited greatly from conversations with R. Bowen, M. W. Hirsch and R. F. Williams.

I. **The genericity problem for diffeomorphisms.** The idea here was to isolate some of the useful properties of Smale's Axiom A and no-cycle diffeomorphisms (see [**22**]) which are not generic [**1**]. The first approach to the problem was fine filtrations.

Recall that for $f \in \mathrm{Diff}^r(M)$ the nonwandering set of f, $\Omega(f)$ or just simply Ω, is the set $\{x \in M \mid$ given any neighborhood U of x there exists an $n > 0$ such that $f^n(U) \cap U \neq \varnothing\}$. Ω contains all periodic orbits and all α and ω limit points. Given $x \in M$, $\alpha(x) = \{y \in M \mid \exists n_i \to -\infty$ and $f^{n_i}(x) \to y\}$ is the α limit set of x and the ω limit set of x, $\omega(x) = \{y \in M \mid \exists n_i \to \infty$ and $f^{n_i}(x) \to y\}$. So Ω is a closed invariant set which contains all the asymptotic behavior of f. If Ω is finite, it consists of periodic points alone.

A filtration for $f \in \mathrm{Diff}^r(M)$, $\mathscr{M}$, is a sequence of compact submanifolds with boundary $\varnothing \subset M_0 \subset M_1 \subset \cdots \subset M_k = M$ with $\dim M_i = \dim M = m$ and $f(M_i) \subset \mathrm{Interior}\, M_i$. Given $\mathscr{M}$, $K_\alpha(\mathscr{M}) = \bigcap_{n \in \mathbf{Z}} f^n(\overline{M_\alpha - M_{\alpha-1}})$ is the maximal f invariant set in $\overline{M_\alpha - M_{\alpha-1}}$ and $K(\mathscr{M}) = \bigcup_{\alpha=1}^k K_\alpha(\mathscr{M})$. So $\overline{M_\alpha - M_{\alpha-1}}$ traps K_α and in principle one should be able to find out a fair amount of information about the complicated set K_α in terms of M_α, $M_{\alpha-1}$, and f. For instance $\check{H}^*(K_0(\mathscr{M})) = \mathrm{limit}(H^*(M_0) \xrightarrow{f^*} H^*(M_0))$.

PROBLEM 1. *Given a finite amount of data about M_α, $M_{\alpha-1}$, and f, compute $\check{H}^*(K_\alpha(\mathscr{M}))$.*

In general, filtrations do not say very much. For example, $\varnothing \subset M$ is a filtration for any diffeomorphism. For any filtration, $\mathscr{M}$, $K(\mathscr{M}) \supseteq \Omega$. A filtration is called fine if $K(\mathscr{M}) = \Omega$. Smale's Axiom A and no-cycle diffeomorphisms have fine filtrations as do many others. An example of

Newhouse [9] showed that the diffeomorphisms which have fine filtrations are not generic. So if Ω cannot be trapped with a single filtration generically how about with a sequence of filtrations?

A filtration $\mathscr{N}=\varnothing\subset N_0\subset\cdots\subset N_j=M$ refines $\mathscr{M}$ if each $\overline{N_\beta-N_{\beta-1}}\subset\overline{M_\alpha-M_{\alpha-1}}$ for some α. A sequence of filtrations $\mathscr{M}^i$ is called fine, $0\leqq i<\infty$, if $\mathscr{M}^{i+1}$ refines $\mathscr{M}^i$ and $\bigcap_{i=0}^\infty K(\mathscr{M}^i)=\Omega$.

CONJECTURE 1. *The diffeomorphisms which have fine sequences of filtrations are generic in* $\mathrm{Diff}^r(M)$.

A phenomenological characterization of those diffeomorphisms which have a fine sequence of filtrations may be given in terms of Ω explosions. $f\in\mathrm{Diff}^r(M)$ admits no C^j Ω explosions, $0\leqq j\leqq r$, if given a neighborhood $U_{\Omega(f)}$ of $\Omega(f)$ in M there is a neighborhood U_f in $\mathrm{Diff}^r(M)$ with the C^j topology such that for $g\in U_f$, $\Omega(g)\subset U_{\Omega(f)}$.

THEOREM 1.1. *$f\in\mathrm{Diff}^r(M)$ has a fine sequence of filtrations if and only if f admits no C^0 Ω explosions.*

The proof of this theorem is in [18] except for the case $m=2$ which is taken care of in [11]. If C^1 Ω explosions replaced C^0 Ω explosions in the theorem, the conjecture would be true in $\mathrm{Diff}^1(M)$ by Pugh's theorem [13].

II. **The genericity problem for vector fields.** The most natural way to state the genericity conjecture for vector fields is in terms of Lyapunov functions. Recall that $L:M\to R$ is a Lyapunov function for $X\in\mathscr{X}^r(M)$, the space of C^r vector fields on M, if and only if $DL=0$ on Ω_X and $X(L)<0$ otherwise, where $X(L)=DL(X(m))$.

CONJECTURE 2. *A generic set of vector fields in $\mathscr{X}^r(M)$ have C^m Lyapunov functions.*

A vector field $X\in\mathscr{X}^r(M)$ generates a one parameter group of diffeomorphisms $\phi_t:M\to M$ defined by

$$d\phi_t(x)/dt\big|_{t=0} = X(x).$$

A filtration $\mathscr{M}$ for a vector field X is a sequence of compact submanifolds with boundary $\varnothing\subset M_0\subset M_1\subset\cdots\subset M_k=M$ such that $\mathscr{M}$ is a filtration for each ϕ_t, $t>0$ and, moreover, $\phi_t(x)$ is transverse to the boundary of M_i for all $x\in M$ and all $i=1,\cdots,k-1$.

The nonwandering set $\Omega_X=\{x\in M\mid$ given a neighborhood $U_x\subset M$ of x and a $T>0$ there exists a $t>T$ with $\phi_t(U_x)\cap U_x\neq\varnothing\}$. A filtration is called fine if

$$K(\mathscr{M}) = \bigcup_{\alpha=1}^{k} K_\alpha(\mathscr{M}) = \bigcup_{\alpha=1}^{k}\left(\bigcap_{t=-\infty}^{\infty}\phi_t(\overline{M_\alpha - M_{\alpha-1}})\right) = \Omega_X.$$

A sequence of filtrations $\mathscr{M}^i$, $0 \leqq i < \infty$, where $\mathscr{M}^{i+1}$ refines $\mathscr{M}^i$ is called fine if $\bigcap_{i=0}^{\infty} K(\mathscr{M}^i) = \Omega_X$. We, of course, have the corresponding:

PROBLEM 2. *Given a finite amount of data about M_α, $M_{\alpha-1}$, and ϕ_t calculate $\check{H}^*(K_\alpha(\mathscr{M}))$.*

A vector field $X \in \mathscr{X}^r(M)$ has no C^j Ω explosions if given a neighborhood U of Ω_X in M, there is a C^j neighborhood V of X in $\mathscr{X}^r(M)$ such that if $Y \in V$ then $\Omega_Y \subset U$.

THEOREM 2.1. *Let $X \in \mathscr{X}^r(M)$. The following are equivalent:*
(a) *X has a fine sequence of filtrations;*
(b) *X has no C^0 Ω explosions;*
(c) *X has a C^m Lyapunov function;*
(d) *X has a C^∞ Lyapunov function.*

PROOF. That (a) is equivalent to (b) is the content of **[11]**. Now, given a fine sequence of filtrations $\mathscr{M}^i$ for X, fix i and then find a C^∞ function $L_i: M \to R$ such that $DL_i = 0$ on $K(\mathscr{M}^i)$, $X(L_i) < 0$ otherwise and $L_i(K_\alpha(\mathscr{M}^i)) = \alpha$ by Proposition 6 of **[11]** and **[14]**. Now choose constants $c_i > 0$ such that $\sum c_i L_i$ is a C^∞ function. So (a) implies (d) which obviously implies (c). To see that (c) implies (a), let $L: M \to R$ be a C^m Lyapunov function for X. By Sard's theorem, the critical values of L are a closed nowhere dense subset of a closed interval. Let $A_n = \{x_0, \cdots, x_{n_k}\}$ be a partition of the interval with mesh $\leqq 1/n$ and the x_i regular values of L. Taking $M_i^n = L^{-1}(-\infty, x_i]$, $i = 1, \cdots, n_k$, and $M_{n_k+1}^n = M$ we have a filtration $\mathscr{M}^n$ for X. Refining A_n to A_{n+1} refines $\mathscr{M}^n$ and we have a fine sequence.

Of course, once again by Pugh's theorem **[13]** if C^1 could replace C^0 in the theorem the genericity conjecture would be true in $\mathscr{X}^1(M)$.

The Lyapunov function approach to the genericity problem has the advantage that it allows one to become even more rhapsodic, by hoping (a) that the vector fields with C^∞ Lyapunov functions are open and dense in $\mathscr{X}^r(M)$ or $\mathscr{X}^\infty(M)$. One may then hope that (b) the Lyapunov function may be selected in a locally smooth fashion for an open and dense set of vector fields and (c) that the Thom-Mather theory of topological stability of mappings may hold in the sense that those vector fields with Lyapunov functions that are topologically stable for the local selection functions are open and dense. So, (d) the vector fields which are topologically Ω-stable are open and dense (see [7]). Recall that X is topologically Ω-stable if and only if there is a neighborhood of X, U_X, in $\mathscr{X}^r(M)$ such that if $Y \in U_X$ then Ω_Y is homeomorphic to Ω_X. Finally, one may hope that (e) for an open and dense set of those vector fields having Lyapunov functions, the flow ϕ_t has a dense orbit in $\Omega \cap L^{-1}(c)$ for any critical value c of L.

While this list of properties is rather outrageous, there is so far no counterexample to my knowledge. There are also precious few examples outside of Smale's Axiom A and no-cycles vector fields and a class of examples which can be derived from [7]. If the program could be carried out a rather beautiful picture would emerge for almost all vector fields. They would have a topography. All the recurrence would be on the ridges, each ridge held together by a dense orbit, all other orbits would flow downhill and the topography would essentially remain unchanged under small perturbations of the vector field.

In the end one might even hope for a general approach to the bifurcation problem for vector fields in terms of the bifurcations of the Lyapunov functions. I suppose that I have fantasized enough. In the next sections I will return to facts and theorems.

III. **A C^0 density theorem for diffeomorphisms.** If one is going to make models of structurally stable or simplest diffeomorphisms in each isotopy class of diffeomorphisms, the first obvious question is: Does every isotopy class of diffeomorphisms contain a structurally stable diffeomorphism? This question was essentially answered in the affirmative by Smale [**23**]. In [**17**] and [**19**], Smale's procedure is elaborated and analyzed some more to produce a C^0 dense set of structurally stable diffeomorphisms in each isotopy class, the nonwandering sets of which are easily describable by a finite collection of matrices which are closely connected to the homology theory of the map. The results in this section are entirely contained in [**23**], [**17**], and [**19**].

In order to produce structurally stable diffeomorphisms, we will produce Axiom A and strong transversality diffeomorphisms. Recall that $f \in \mathrm{Diff}^r(M)$ satisfies Smale's Axiom A if and only if

(a) $\Omega(f)$ has a hyperbolic structure;

(b) $\Omega(f)$ is the closure of the periodic points of f.

That $\Omega(f)$ has a hyperbolic structure means that $TM|\Omega(f)$, the tangent bundle of M restricted to $\Omega(f)$, may be written as the direct sum of two Tf invariant subbundles $E^s \oplus E^u$ such that there exist constants $0<\lambda<1$, $0<C$, and

$$\|Tf^n \mid E^s\| \leqq C\lambda^n \quad \text{for } n > 0;$$
$$\|Tf^n \mid E^u\| \leqq C\lambda^n \quad \text{for } n < 0.$$

If $f \in \mathrm{Diff}^r(M)$ and $x \in M$,

$$W^s(x) = \{y \in M \mid d(f^n(x), f^n(y)) \to 0 \text{ as } n \to \infty\}, \quad \text{and}$$
$$W^u(x) = \{y \in M \mid d(f^n(x), f^n(y)) \to 0 \text{ as } n \to -\infty\}.$$

If f satisfies Axiom A then $W^s(x)$ and $W^u(x)$ are 1:1 immersed Euclidean spaces for all $x \in M$. If, moreover, $W^s(x)$ and $W^u(x)$ are transversal for all $x \in M$, then f is said to satisfy the strong transversality condition. Joel Robbins has proven:

THEOREM **[15]**. *Let $f \in \mathrm{Diff}^1(M)$ be C^2. If f satisfies Axiom* A *and the strong transversality condition then f is structurally stable.*

Recently, Clark Robinson has removed the condition that f be C^2. But for our purposes C^2 is sufficient because we can perturb f to be C^∞.

The first step in isotoping f to a structurally stable diffeomorphism is to make it preserve a special filtration, a handle decomposition of M. Recall that a handle decomposition, $\mathscr{H}$, of M is a sequence of submanifolds $\varnothing \subset M_0 \subset \cdots \subset M_m = M$ where $\overline{M_j - M_{j-1}} = \bigcup_{i=1}^{n_j} (D_i^j \times D_i^{m-j})$ and the $D_i^j \times D_i^{m-j}$ are attached to the boundary of M_{j-1} by disjoint embeddings. If $\mathscr{H}$ is a filtration for f we will say $f \in T_{\mathscr{H}}$ if, moreover, $f(D_i^j \times 0)$ is transverse to $0 \times D_k^{m-j}$ for $1 \leqq i$, $k \leqq n_j$ and for all j. If $f \in T_{\mathscr{H}}$, $f(D_i^j \times 0)$ intersects $0 \times D_k^{m-j}$ in a finite number, g_{ik}^j, of points. We may form the geometric intersection matrix $G_j = (g_{ik}^j)$. We may also form the algebraic intersection matrix $A_j = (a_{ik}^j)$ where a_{ik}^j is the number of points of intersection of $f(D_i^j \times 0)$ with $0 \times D_k^{m-j}$ counted with their signs. A_j may have negative entries but clearly $|a_{ik}^j| \leqq g_{ik}^j$. The A_j's determine an endomorphism of the complex

$$\begin{array}{ccc} \cdots \to H_j(M_j, M_{j-1}) & \xrightarrow{\partial} & H_{j-1}(M_{j-1}, M_{j-2}) \to \cdots \\ \big\downarrow{\scriptstyle A_j} & & \big\downarrow{\scriptstyle A_{j-1}} \\ \cdots \to H_j(M_j, M_{j-1}) & \xrightarrow{\partial} & H_{j-1}(M_{j-1}, M_{j-2}) \to \cdots \end{array}$$

which induces $f_*: H_*(M) \to H_*(M)$ on homology. The G_j's will be used to determine the nonwandering sets. If $\mathscr{H}$ is a filtration for f we let $\Omega_j = \Omega \cap (M_j - M_{j-1})$. $\Omega_j \subseteq K_j$.

DEFINITION. The subset $H \subset \mathrm{Diff}^r(M)$ is the subset of those diffeomorphisms $f \in \mathrm{Diff}^r(M)$ such that:

(1) f satisfies Axiom A and the strong transversality condition;

(2) $f \in T_{\mathscr{H}}$ for some handle decomposition $\mathscr{H}$ of M;

(3) $f|K_j$ is topologically conjugate to the subshift of finite type associated to the geometric intersection matrix G_j and $f|\Omega_i$ is topologically conjugate to the subshift of finite type restricted to its nonwandering set.

I will now explain (3). Note that the diffeomorphisms in H are structurally stable. I have used H to denote Smale's handle preserving, horseshoe diffeomorphisms; the difference here from Smale's **[23]** is the strong transversality condition and the identification of the $f|K_i$ in terms of the G_i. Let $B = (b_{ij})$ be an $n \times n$ matrix. B is a 0-1 matrix if $b_{ij} = 0$ or 1 for all i, j. Let $N=(1, \cdots, n)$ with the discrete topology and $\Sigma=\prod_{-\infty}^{\infty} N$ have the product topology. The elements of Σ are of the form $\{a_i\}_{i\in Z}$ where $a_i \in N$. The shift map $\sigma:\Sigma\to\Sigma$ is defined by $\sigma(\{a_i\}_{i\in Z})=\{a_i'\}_{i\in Z}$ where $a_i'=a_{i+1}$. The 0-1 matrix B defines a σ invariant closed subset of Σ, Σ_B, by $\{a_i\}_{i\in Z}\in\Sigma_B$ if and only if $b_{a_i a_{i+1}}=1$. $\sigma:\Sigma_B\to\Sigma_B$ is a subshift of finite type. Now given a nonnegative $n\times n$ matrix G with integral entries we may associate a 0-1 matrix B to it as follows. Consider $G:Z^n\to Z^n$ to be a linear map where Z^n has standard basis $e_1, \cdots, e_n$. Let $d_i=\sum_{k=1}^n g_{ik}$, and let $m=\sum_i d_i$. B will be an $m\times m$ matrix and we think of it as operating on Z^m. We consider Z^m to be generated by

$$e_{11}, \cdots, e_{1d_1}, e_{21}, \cdots, e_{2d_2}, \cdots, e_{n1}, \cdots, e_{nd_n}$$

and define

$$b_{ij,ml}=\begin{cases} 1 & \text{if } \sum_{k=1}^{m-1} g_{ik} < j \leqq \sum_{k=1}^{m} g_{ik}, \\ 0 & \text{otherwise.}\end{cases}$$

B is the 0-1 matrix associated to G and $\sigma:\Sigma_B\to\Sigma_B$ is the subshift of finite type associated to G. Now the assertion in (3) is that there is a surjective homeomorphism h mapping K_i onto Σ_B such that $hf=\sigma h$, etc.

THEOREM 3.1. *Any $f\in \mathrm{Diff}^r(M)$ is isotopic to an element of H by a C^0 small isotopy. So H is C^0 dense in* $\mathrm{Diff}^r(M)$.

Given $f\in H$ we may calculate the number of periodic points of f of period m, $N_m(f)$, via the geometric intersection matrices G_i.

PROPOSITION 3.2. *Let $f\in H$. Then $N_m(f)=\sum_i$ trace G_i^m.*

We may in fact compute a sort of asymptotic Lefschetz inequality for diffeomorphisms in H.

PROPOSITION 3.3. *Let $f\in H$. Then*

$$\limsup n^{-1}\log N_n(f) \geqq \max \log|\lambda|$$

where the max *is taken over all eigenvalues of $f_*:H_*(M, Q)\to H_*(M, Q)$.*

The reason for this is that the spectral radius of G_i is bigger than or equal to the spectral radius of the A_i (because $|a_{ik}^j|\leqq g_{ik}^j$; see [**6**]), which is bigger than or equal to the spectral radius of the $f_{*i}:H_i(M, Q)\to H_i(M, Q)$, and in the trace formula of 3.2 there is no alteration of signs! I will return to this point later.

The proof of Theorem 3.1 proceeds via fitting the diffeomorphism. Given the handle decomposition $\mathscr{H}$ of M, call any disc of the form $D_j^i\times q$ where $q\in D_j^{m-i}$ a core disc, f is fitted with respect to $\mathscr{H}$ if f (core disc) contains any core disc it intersects. Using standard techniques of differential topology and a double induction, any diffeomorphism may be isotoped to one which is fitted. This process essentially proves 3.1 by pulling along the core discs to get an expansion in the core disc direction and contracting along the transverse discs to assure the hyperbolic structure. So Theorem 3.1 may be modified to say that any f is isotopic to a fitted element of H by a C^0 small isotopy.

In some sense, fitted diffeomorphisms are the analogue for diffeomorphisms of handle decompositions for differentiable manifolds. I will pursue the analogue of Smale's theorem on the structure of manifolds to try to trace the simplest matrices that can occur for the G_i's. Recall that we are limited by having the A_i's, which are an endomorphism of a chain complex which arises from a handle decomposition of M and which gives the homology of M. $g^i_{jk} \geqq |a^i_{jk}|$ so the most efficient picture would be given by $g^i_{jk} = |a^i_{jk}|$. If E is a matrix, $|E|$ denotes the matrix whose entries are the absolute values of the entries of E.

In what follows a free chain complex will mean a free chain complex $\mathscr{C}$, $0 \to C_m \to C_{m-1} \to \cdots \to C_2 \to C_1 \to C_0 \to 0$, where $m = \dim M$. Henceforth, M will be assumed to be connected.

THEOREM (SMALE [**20**]). *Let $\Pi_1(M)=0$ and $\dim M \geqq 6$. If $\mathscr{C}$ is a free chain complex with $C_1 = 0 = C_{m-1}$ such that $H_k(M, Z) = H_k(C, Z)$ for all k, then $\mathscr{C}$ is the free chain complex of a handle decomposition of M.*

Such a complex will be called a complex of M. The next theorem identifies a class of matrices G_i which may occur as the geometric intersection matrices of a diffeomorphism isotopic to f. It is stated somewhat differently than in [**19**]. Note that the hypothesis amounts to chain homotopy.

THEOREM 3.4. *Let $\Pi_1(M)=0$ and $\dim M \geqq 6$. Suppose that $f \in \mathrm{Diff}^r(M)$ and that $\mathscr{E}$ is an endomorphism of a chain complex $\mathscr{C}$ of M given as matrices E_i. If*

$$f_* : H_*(M, Z) \to H_*(M, Z) = \mathscr{E}_* : H_*(\mathscr{C}, Z) \to H_*(\mathscr{C}, Z)$$

and

$$f_* : H_*(M, Z_n) \to H_*(M, Z_n) = \mathscr{E}_* : H_*(\mathscr{C}, Z_n) \to H_*(\mathscr{C}, Z_n)$$

for all n, then f is isotopic to a fitted diffeomorphism in H with $G_i = |E_i|$.

This theorem reduces the problem of determining a large class of models of structurally stable diffeomorphisms to a purely algebraic problem. It is the main tool for the next section. Such an $\mathscr{E}$ will be called an endomorphism of f.

IV. Morse-Smale diffeomorphisms. The Morse-Smale diffeomorphisms are the simplest diffeomorphisms of all. They are the Axiom A and strong transversality diffeomorphisms with a finite Ω. So their nonwandering sets consist of a finite number of periodic orbits. Palis and Smale proved:

THEOREM [**12**]. *The Morse-Smale diffeomorphisms are precisely the structurally stable diffeomorphisms with a finite* Ω.

In 1967 Smale [**21**] raised the question of which diffeomorphisms are isotopic to Morse-Smale diffeomorphisms. In the summer of 1971, I realized that a Morse-Smale diffeomorphism had to be quasi-unipotent on homology; that is, every eigenvalue of $f_*: H_*(M, Q) \to H_*(M, Q)$ must be a root of unity or equivalently $\max \log|\lambda| = 0$ where the max is taken over all eigenvalues of f_*. But actually more can be said. The matrix

$$\begin{pmatrix} P_1 & * & \cdots & & * \\ \cdot & P_2 & & & \cdot \\ \cdot & & \cdot & & \cdot \\ \cdot & & & \cdot & \cdot \\ 0 & & & & P_n \end{pmatrix}$$

where each P_i is a signed permutation matrix (with ± 1's) is called a virtual permutation.

THEOREM 4.1. *If $f \in \mathrm{Diff}^r(M)$ is Morse-Smale, then there is a finite length chain complex $\mathscr{C}$ of finitely generated abelian groups $\cdots \to C_{i+1} \xrightarrow{\partial} C_i \to \cdots$ with a chain automorphism $\mathscr{F} = \{C_i \xrightarrow{F_i} C_i\}$ so that*

(1) *the F_i are virtual permutation matrices;*

(2) *the pair $(\mathscr{C}, \mathscr{F})$ is equivalent to a geometric chain map induced by f.*

For simply connected manifolds of high dimension, we can give necessary and sufficient conditions.

THEOREM 4.2. *Let $\Pi_1(M) = 0$ and $\dim M \geqq 6$. Then $f \in \mathrm{Diff}^r(M)$ is isotopic to a Morse-Smale diffeomorphism if and only if there is an endomorphism $\mathscr{E}$ of f such that E_i is a virtual permutation for all i.*

Now by a theorem of Bowen [**2**], the entropy of a map is equal to the entropy of that map restricted to its nonwandering set. So the entropy of a Morse-Smale diffeomorphism is zero. Consequently, at least for the isotopy classes described in Theorem 4.2, we can construct a simplest diffeomorphism, in fact, a fitted element of H.

PROBLEM 3. *Are the only structurally stable diffeomorphisms with zero entropy the Morse-Smale diffeomorphisms?*

A positive answer to this problem would justify calling the Morse-Smale diffeomorphisms the simplest of all in the sense of simplest above.

A negative answer would be rather startling because it would contradict the conjecture of Palis and Smale that the structurally stable diffeomorphisms are precisely the Axiom A and strong transversality diffeomorphisms.

Theorem 4.2 may also be stated as follows: The homology class of graph f can be constructed by a virtual permutation of a chain complex for the manifold. Finding such a virtual permutation of a chain complex of course implies that f_* is quasi-unipotent. But the converse is not true.

THEOREM 4.3. *Let* $\Pi_1(M)=0$ *and* $\dim M \geqq 6$. *Let* $f \in \mathrm{Diff}^r(M)$. *Then there exists an* $n>0$ *such that* f^n *is isotopic to a Morse-Smale diffeomorphism if and only if* $f_*: H_*(M, Q) \to H_*(M, Q)$ *is quasi-unipotent.*

The reason we have to take a finite power and cannot get by with f itself is because of an obstruction closely related to the ideal classes of the cyclotomic fields. This obstruction is nonzero, according to R. G. Swan.

To give an example, suppose $H_*(M)$ is torsion free. If $f_{*i}: H_i(M) \to H_i(M)$ may be put in the form

$$\begin{pmatrix} A_1 & * & \cdots & * \\ \cdot & \cdot & & \cdot \\ \cdot & & \cdot & \cdot \\ \cdot & & & * \\ 0 & \cdot & \cdot & A_n \end{pmatrix}$$

where each A_i is equivalent over Z to the companion matrix of its characteristic polynomial then f is isotopic to a Morse-Smale diffeomorphism. Of course, I am assuming here that $\Pi_1(M)=0$ and $\dim M \geqq 6$. The obstruction is analyzed somewhat more in [**19**].

V. **Entropy.** For Axiom A diffeomorphisms Bowen [**2**] proved:

THEOREM. $h(f) = \limsup n^{-1} \log N_n(f)$.

So it now follows from the asymptotic Lefschetz inêquality of Proposition 3.3 that if $f \in H$, then

$$(*) \qquad h(f) \geqq \max \log |\lambda|$$

where the max is taken over all eigenvalues of $f_*: H_*(M, Q) \to H_*(M, Q)$. The main problem I want to consider here is the extent to which $(*)$ holds. Combining the density theorem with Nitecki [**10**] gives:

THEOREM 5.1. $(*)$ *holds for an open and dense set of* $\mathrm{Diff}^r(M)$ *in the* C^0 *topology.*

In fact, it seems fairly clear that $\mathrm{Diff}^r(M)$ may be replaced by $\mathrm{End}^r(M)$ the space of C^r endomorphisms of M.

CONJECTURE 3. (a) (*) *holds for all smooth mappings of compact differentiable manifolds*;

(b) (*) *holds for all Axiom* A *and strong transversality diffeomorphisms.*

Conjecture 3(a) of course implies (b), but there is more evidence for (b). Bowen, for example, has proven (b) when Ω is 0-dimensional. Conjecture 3(a) would be a rather striking theorem, generalizing in a sense I will describe below the Lefschetz fixed point theorem. Conjecture 3(b) is rather important in its own right if only for the restrictions it imposes on simplest diffeomorphisms.

The Lefschetz number of f^n is given by $L(f^n)=\sum (-1)^i \operatorname{trace} f^n_{*i}$ so we may form the asymptotic Lefschetz number of f.

$$l(f) = \lim\sup n^{-1} \log|L(f^n)|.$$

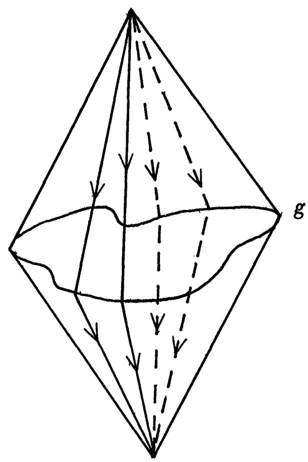
If λ is an eigenvalue of largest modulus for f_* and all other eigenvalues of the same modulus occur in dimensions of the same parity, then $l(f)=\log|\lambda|$. In other words, the asymptotic Lefschetz number is $\log|\lambda|$ unless there is some cancelling out in the trace formula due to the alteration of signs.

$L(f^n)=\sum_{P\in\mathrm{Fix}\, f^n} \sigma(P)$ where σ is the index of P, if all the fixed points of f^n are isolated. So the Lefschetz number counts the fixed points of f^n with their indices. This raises an immediate problem about $l(f)$ itself.

PROBLEM 4. *Suppose $f:M\to M$ is smooth. If the fixed points of f^n are isolated for all n, is* $\lim\sup n^{-1} \log N_n(f)\geqq l(f)$?

This is true for Kupta-Smale endomorphisms which are a generic set in $\mathrm{End}^r(M)$ (see **[16]**). Conversations I have had with Dennis Sullivan make it seem very likely (almost a proven theorem) that if $l(f)>0$ then $N_n(f)\to\infty$. So the asymptotic Lefschetz number should give an estimate of the asymptotic growth of the periodic orbits, but suffers from the alternation of signs. If we drop the alternation of signs in the formula, we get max $\log|\lambda|$ all the time and hopefully estimate the asymptotic growth rate of the orbits, i.e., the entropy. For example: start with any diffeomorphism $f:M\to M$, consider $f\times\theta:M\times S^1\to M\times S^1$ where θ is an irrational rotation of S^1. Then $l(f)=0$ and there are no-periodic points. But $h(f\times\theta)=h(f)$, so if f satisfies (*) so does $f\times\theta$.

Problem 4 and (*) both fail for homeomorphisms of complexes and continuous maps of manifolds, so the smoothness seems crucial. Given $f:X\to X$ we can construct the suspension of f, $S(f):S(X)\to S(X)$. Now compose $S(f)$ on the left with a map which pushes down from the north pole of the suspension to the south pole, g.



The new nonwandering set is just the north and south poles which are the only periodic points. On the other hand $l(g \circ f)=l(f)$ and max $\log|\lambda|$ is unchanged.

There is something which may be said about continuous endomorphisms of finite complexes which follow directly from Franks [5].

PROPOSITION 5.2. *Let K be a finite complex, and let $f:K\to K$ be continuous and have a fixed point. Let $Z^k\subset H^1(K)$ be invariant for f^n. Suppose that $\rho_1, \cdots, \rho_k \in Z^k$ and $\rho_1 \cup \cdots \cup \rho_k \neq 0$. Suppose also that $f^*:Z^k\to Z^k$ has no eigenvalue of absolute value one. Then $h(f)\geqq\sum\log|\lambda|$ where the sum is taken over all eigenvalues λ of $f^*:Z^k\to Z^k$ with $|\lambda|>1$.*

The most natural place to apply this proposition is to the n-torus, T^n. There one sees that those elements A of $SL(n, z)$ which have no eigenvalues of absolute value one (the linear Anosov diffeomorphisms) are simplest diffeomorphisms.

Finally, I would like to return these considerations to the problem of finding simplest diffeomorphisms. The first problem here is to determine if the elements of H will provide simplest diffeomorphisms if they do exist in an isotopy class.

PROBLEM 5. *If f is structurally stable (or Axiom* A *and strong transversality), does there exist a g isotopic to f so that $g \in H$ and $h(g)\leqq h(f)$?*

Bowen [3] and Manning [8] are very relevant here. An affirmative answer to this problem would, of course, prove Conjecture 3(b).

PROBLEM 6. *Let $\mathcal{J}$ be an isotopy class in* $\mathrm{Diff}^r(M)$. *When does there exist a $g \in H\cap\mathcal{J}$ so that equality holds in $(*)$ for g?*

If $\max \log|\lambda| = 0$, such a g would have to be Morse-Smale by Bowen [**2**]. So the theorem above about Morse-Smale diffeomorphisms is a first step in this direction. There was an ideal class problem there. It is also possible that equality cannot be achieved for more simple reasons. Let M be the connected sum of $S^3 \times S^3$ with itself four times

$$M = S^3 \times S^3 \# S^3 \times S^3 \# S^3 \times S^3 \# S^3 \times S^3.$$

M is an $n-1$ connected $2n$ manifold so we may apply Wall's theory [**24**]. $H_3(M) = \bigoplus_8 Z$ and is the only relevant group for our problem. By Wall's theory given an automorphism A of $H_3(M)$ there will be a diffeomorphism $\hat{A}$ of M which induces this automorphism if A preserves the intersection matrix, K, on $H_3(M)$ which is the 8×8 matrix

$$K = \begin{pmatrix} 01 & & \cdot & \cdot & \cdot & & 0 \\ -10 & & & & & & \\ & & 01 & & & & \cdot \\ \cdot & & -10 & & & & \cdot \\ \cdot & & & & 01 & & \cdot \\ \cdot & & & & -10 & & \\ & & & & & & 01 \\ 0 & & \cdot & \cdot & \cdot & & -10 \end{pmatrix}$$

which may be put in the form $K' = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$ over Z.

In this new basis we are looking for a matrix A' such that $A'^t K A' = K$. But now any matrix of the form

$$\begin{pmatrix} B & 0 \\ 0 & (B^{-1})^t \end{pmatrix}$$

will do. Choose B to be the companion matrix of the polynomial $X^4 + X + 1$. Now, some Galois theory (for which I am indebted to Nick Bourgoyne and Marvin Greenberg) will show that (1) not all of the roots of this polynomial are roots of unity times a real number, and as is easily seen (2) all the roots are complex.

Now having constructed an element of H, g, isotopic to the $\hat{A}$ produced in this way, we have

$$C_3 \xrightarrow{A_3} C_3 \quad \text{and} \quad H_3(m) \xrightarrow{A} H_3(M)$$

and $|a_{ij}^3| \leqq g_{ij}^3$ for all i, j. By the theory of nonnegative matrices [**6**], if the maximum modulus of an eigenvalue of G_3 equals the maximum modulus of an eigenvalue of A_3 then all the eigenvalues of A_3 of this modulus are real numbers times roots of unity. Since the eigenvalues of A are eigenvalues of A_3, the choice of A implies that the spectral radius of G_3 is strictly bigger than the spectral radius of A. But by the above for any $g \in H$

$$\begin{aligned} h(g) &= \limsup n^{-1} \log N_n(g) \\ &= \limsup n^{-1} \log\left(\sum \operatorname{trace} G_i^n\right) \\ &= \max \log |\lambda|, \end{aligned}$$

where the max is taken over all the eigenvalues of the geometric intersection matrices G_i for g. This proves that for any $g \in H$ which is isotopic to A, $h(g) > \max \log|\lambda|$. So the difficulty in finding an element of H which achieves equality in (*) lies at least in the problems of the ideal classes and in the problem of increasing the eigenvalues of a matrix when the absolute value is taken (this is actually how the ideal classes came up in the first place). Neither of these connections seems to be well understood.

If the answer to Problem 5 is yes, there will be isotopy classes for which an affirmative answer to the following problem would indicate the non-existence of a single simplest diffeomorphism. One would have to use a sequence.

PROBLEM 7. *Let $\mathscr{I}$ be an isotopy class in* $\mathrm{Diff}^r(M)$; *does there exist a sequence of diffeomorphisms $f_n \in \mathscr{I} \cap H$ such that $h(f_n) \to \max \log|\lambda|$ where the* max *is taken over all eigenvalues of $f_* : H_*(M, Q) \to H_*(M, Q)$?*

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