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A COHOMOLOGY FOR FOLIATED MANIFOLDS

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1. **Introduction.** Let M be a connected manifold and τ a foliation on M. τ is then an involutive subbundle of TM, the tangent bundle of M. Denote by ν the normal bundle to τ, $\nu = TM/\tau$. We denote sections of a bundle P over M by $\Gamma(P)$. All manifolds, bundles and maps are assumed to be C^∞.

There is a canonical connection ∇ on ν which is flat along τ [**B**]. Consider the complex

$$\Gamma(\nu) \xrightarrow{\hat{d}} \Gamma(\nu \otimes \Lambda^1\tau^*) \xrightarrow{\hat{d}} \Gamma(\nu \otimes \Lambda^2\tau^*) \xrightarrow{\hat{d}} \cdots,$$

where τ^* is the cotangent bundle to the foliation and

$$\begin{aligned}\hat{d}(\sigma)(X_1, \ldots, X_{k+1}) &= \sum_{1 \leq i \leq k+1} (-1)^i \nabla_{X_i}(\sigma(X_1, \ldots, \hat{X}_i, \ldots, X_{k+1})) \\ &+ \sum_{1 \leq i < j \leq k+1} (-1)^{i+j+1} \sigma([X_i, X_j], X_1, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_{k+1})\end{aligned}$$

for $\sigma \in \Gamma(\nu \otimes \Lambda^k\tau^*)$, $X_1, \ldots, X_{k+1} \in \Gamma(\tau)$.

Since the curvature tensor of ∇ restricted to τ is identically zero we have that $\hat{d} \circ \hat{d} = 0$. Denote the homology of this complex by $F^*(\tau; \nu)$. This is the cohomology of the Lie algebra of vector fields tangent to the foliation with coefficients in sections of the normal bundle, the representation being given by the connection [**GF**].

In general the groups $F^k(\tau; \nu)$ are not finitely generated (the complex is not elliptic) but they satisfy the following.

(i) F^* is a functor from the category of foliated manifolds and transverse maps to the category of abelian groups and homomorphisms.

(ii) If $f: N \to M$ is an embedded transverse submanifold, we can define relative cohomology groups $F^*(\tau; \nu, f)$ and obtain the usual long exact sequence.

(iii) F^* is an invariant of the diffeomorphism type of the foliation. However, F^* is not an invariant of the integrable homotopy type of the foliation when M is an open manifold.

2. **Interpretation of $F^1(\tau; \nu)$.** Fix a Riemannian metric on M and think

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of ν as all tangent vectors normal to τ. We then have projection operators $\pi: TM \to \tau$ and $\pi^{\perp}: TM \to \nu$. We can view $\Gamma(\nu \otimes \tau^*)$ as infinitesimal deformations of τ as follows: Let τ_s, $s \in \boldsymbol{R}$, be a differentiable family of codimension q subbundles of TM with $\tau_0 = \tau$ and let π_s and $\pi_s^{\perp}$ be the associated projection operators. For each $X \in \tau_0$ define

$$\sigma(X) = d/ds\,(\pi_s(X))|_{s=0}.$$

We call σ the infinitesimal deformation associated to the family τ_s and note that $\sigma \in \Gamma(\nu \otimes \tau^*)$.

For each $s \in \boldsymbol{R}$, let $\Phi_s(X, Y) = \pi_s^{\perp}([\pi_s X, \pi_s Y])$.

LEMMA. (i) *Φ_s is an exterior 2 form on TM.*
(ii) *τ_s is involutive if and only if $\Phi_s \equiv 0$.*
(iii) $\hat{d}\sigma(X, Y) = d/ds\,(\Phi_s(X, Y))|_{s=0}$.

PROPOSITION 1. *Let τ_s be a differentiable family of subbundles of TM all of which are foliations. Let σ be the associated infinitesimal deformation. Then $\hat{d}\sigma = 0$.*

PROPOSITION 2. *Let ϕ_s, $s \in \boldsymbol{R}$, be a flow on M, X the associated vector field and τ_0 a foliation. For each $s \in \boldsymbol{R}$, $(\phi_s)_*\tau_0$ is a foliation on M denoted τ_s. Let σ be the associated infinitesimal deformation. Then $\sigma = \hat{d}(\pi_0^{\perp}X)$.*

Note that any element $X \in \Gamma(\nu)$ generates a local flow whose associated infinitesimal deformation is $\hat{d}X$. Thus we may view $F^1(\tau; \nu)$ as infinitesimal deformations of the foliation τ, modulo trivial deformations.

Question. Given $\alpha \in F^1(\tau; \nu)$, under what conditions does there exist an element $\sigma \in \alpha$ which comes from a deformation of τ through foliations?

EXAMPLE. Constant slope foliations on T^2. Let $T^2 = \boldsymbol{R}^2/\boldsymbol{Z}^2$ and denote by τ_a the foliation of T^2 given by all straight lines of slope a.

THEOREM 1. *If a is rational then $F^0(\tau_a; \nu_a) \cong C^\infty(S^1)$, $F^1(\tau_a; \nu_a) \cong C^\infty(S^1)$ and each element of $F^1(\tau_a; \nu_a)$ can be realized as the associated infinitesimal deformation of a differentiable family of foliations.*

Recall that an irrational number a is *not* a Liouville number provided there is a positive integer p and $\varepsilon > 0$ such that $|a - n/m| > \varepsilon(|m| + |n|)^{-p}$ for m and n sufficiently large integers.

THEOREM 2. *If a is irrational then $F^0(\tau_a; \nu_a) \cong \boldsymbol{R}$. If a is not a Liouville number then $F^1(\tau_a; \nu_a) \cong \boldsymbol{R}$ and each element of $F^1(\tau_a; \nu_a)$ can be realized as the associated infinitesimal deformation of a differentiable family of foliations.*

See [**H**].

3. **The complex restricted to a leaf.** Let L be a leaf of a foliation τ on M

and denote by ν the normal bundle of τ restricted to L. Consider the complex

$$\Gamma(\nu) \xrightarrow{\hat{d}} \Gamma(\nu \otimes \Lambda^1 T^*L) \xrightarrow{\hat{d}} \Gamma(\nu \otimes \Lambda^2 T^*L) \xrightarrow{\hat{d}} \cdots,$$

where $\hat{d}$ is defined as above. Again $\hat{d}^2 = 0$ and we denote the resulting groups by $F^*(L)$.

THEOREM 3. *$F^*(L)$ is isomorphic to $H^*(L; \boldsymbol{R}^q_\sim)$, the cohomology of L with coefficients in $\boldsymbol{R}^q_\sim$ ($q = \dim \nu$) twisted over the linear holonomy of the foliation τ.*

We prove this by noting that the linear holonomy of the foliation is the holonomy of the canonical connection on ν. We then show that the complex $\{\Gamma(\nu \otimes \Lambda^k T^*L), \hat{d}\}$ is isomorphic to the complex $\{\mathscr{A}_{\pi_1(L)}(\tilde{L}; \boldsymbol{R}^z_\sim), d\}$, the de Rham complex of $\boldsymbol{R}^q_\sim$ valued forms on the simply connected covering space $\tilde{L}$ of L which satisfy:

$$(\sigma^*\omega)(Y_1, \ldots, Y_k) = h(\sigma^{-1})(\omega(Y_1, \ldots, Y_k)),$$

where ω is an $\boldsymbol{R}^q_\sim$ valued k-form on $\tilde{L}$, $Y_1, \ldots, Y_k \in \Gamma(T\tilde{L})$, $\sigma \in \pi_1(L)$ and acts on $\tilde{L}$, by deck transformations, and $h: \pi_1(L) \to \mathrm{GL}(q, \boldsymbol{R})$ is the holonomy representation.

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