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TOPOLOGIES ON THE RATIONAL FIELD

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Although the ring topologies on the field $\boldsymbol{Q}$ of rationals defy classification (see [**2**, §1]), we are able to resolve a longstanding problem in the theory of topological rings by showing that the only locally bounded ring topologies on $\boldsymbol{Q}$ are the known ones, and in particular, the only Hausdorff, locally bounded, additively generated topology on $\boldsymbol{Q}$ (a ring topology is additively generated if there are no proper open additive subgroups) is the ordinary archimedean topology.

Let P be the set of prime numbers, and for each $p \in P$ let $|\,.\,.\,|_p$ denote the p-adic absolute value on $\boldsymbol{Q}$. Let $|\,.\,.\,|_\infty$ denote the ordinary archimedean absolute value on $\boldsymbol{Q}$, and let $P' = P \cup \{\infty\}$. For each subset R of P', let $O(R) = \{x \in \boldsymbol{Q} : |x|_p \leqq 1 \text{ for all } p \in R\}$. As is well known, for each subset R of P' there is a unique locally bounded ring topology $\mathscr{T}_R$ on $\boldsymbol{Q}$ for which $O(R)$ is a bounded neighborhood of zero (see [**1**, Exercise 20, pp. 120–121]); if $R \neq P'$, a fundamental system of neighborhoods of zero for $\mathscr{T}_R$ consists of all $O(R)x$, where x is a nonzero rational. Note that $\mathscr{T}_{P'}$ is the discrete topology, and $\mathscr{T}_\varnothing$ is the nonHausdorff ring topology.

THEOREM. *The only locally bounded ring topologies on $\boldsymbol{Q}$ are the topologies $\mathscr{T}_R$, where R is a subset of P'. In particular, the only Hausdorff, locally bounded, additively generated topology on $\boldsymbol{Q}$ is the ordinary archimedean topology $\mathscr{T}_\infty$.*

To prove the Theorem, we first identify the completion of $\boldsymbol{Q}$ for $\mathscr{T}_R$, where R is a nonempty proper subset of P', with the local direct product A_R of the fields $\boldsymbol{Q}_p$ relative to the open subrings $\boldsymbol{Z}_p$, where $p \in R$ ($\boldsymbol{Q}_p$ and $\boldsymbol{Z}_p$ are respectively the field (ring) of p-adic numbers (integers) if p is a prime; $\boldsymbol{Q}_\infty = \boldsymbol{Z}_\infty =$ the real field). The crucial step is to show that if a Hausdorff locally bounded ring topology $\mathscr{T}$ on $\boldsymbol{Q}$ is weaker than $\mathscr{T}_R$ for some proper subset R of P', then $\mathscr{T} = \mathscr{T}_S$ for some proper subset S of P'; this is accomplished by studying the completion of $\boldsymbol{Q}$ for $\mathscr{T}$ as a topological algebra over the topological ring A_R. We then apply two results of Mutylin (the only results known thus far concerning the classification of locally bounded ring topologies on $\boldsymbol{Q}$); the first [**2**, Theorem 2] is that if $\mathscr{T}$ is not stronger than $\mathscr{T}_\infty$, then $\mathscr{T} = \mathscr{T}_S$ for some subset S of P (the above step

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enables us to reduce by half the length of Mutylin's ingenious proof); the second [**2**, Corollary 5] is that if $\mathscr{T}$ is stronger than $\mathscr{T}_\infty$ but not discrete, then $\mathscr{T}$ is weaker than $\mathscr{T}_S$ for some proper subset S of P'.

The following corollaries follow easily; the second generalizes a theorem of Mutylin [**2**, Theorem 3].

COROLLARY 1. *The only nondiscrete locally compact rings containing* $\boldsymbol{Q}$ *densely are the rings* A_R, *where* R *is a nonempty proper subset of* P'.

COROLLARY 2. *If* A *is a Hausdorff, complete, locally bounded ring containing* $\boldsymbol{Q}$, *then either* $\boldsymbol{Q}$ *is discrete, or the closure of* $\boldsymbol{Q}$ *is* A_R *for some nonempty proper subset* R *of* P'; *in particular, if* A *is, in addition, a field, then either* $\boldsymbol{Q}$ *is discrete, or the closure of* $\boldsymbol{Q}$ *is either the real field or the p-adic number field for some prime p.*

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