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ALL OPERATORS ON A HILBERT SPACE ARE BOUNDED

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Introduction. Following Solovay [2], let 'ZF' denote the axiomatic set theory of Zermelo-Fraenkel and let 'ZF + DC' denote the system obtained by adjoining a weakened form of the axiom of choice, DC, (see p. 52 of [2] for a formal statement of DC). From DC a 'countable' form of the axiom of choice is obtainable. More precisely, if $\{B_n : n \in N\}$ is a countable collection of nonempty sets then it follows from DC that there exists a function f with domain N such that $f(n) \in B_n$ for each n.

The system ZF + DC is important because all the positive results of elementary measure theory and most of the basic results of elementary functional analysis, except for the Hahn-Banach theorem and other such consequences of the axiom of choice, are provable in ZF + DC. In particular, the Baire category theorem for complete metric spaces and the closed graph theorem for operators between Fréchet spaces are provable in ZF + DC.

Solovay shows [2] that the proposition, *Each subset of the real numbers is Lebesgue measurable*, cannot be disproved in ZF + DC. He does this by constructing a model for ZF + DC in which the proposition becomes a true statement.

We shall see that the proposition, *Each linear operator on a Hilbert space is a bounded linear operator*, is consistent with the axioms of ZF + DC. Other results of this type are obtained. For example, *Whenever X and Y are separable Fréchet groups and $h: X \to Y$ is a homomorphism then h is continuous*, cannot be proved or disproved in ZF + DC.

Fortunately all the hard work in model theory has been done by Solovay. All that we use here is straightforward functional analysis.

All operators on a Hilbert space are bounded. We recall that a subset S of a topological space T is said to have the *Baire property* if there exists an open set U such that $(U \setminus S) \cup (S \setminus U)$ is meagre. Let BP be the proposition: *Each subset of a complete separable metric space has the Baire property*. In [2, §4], Solovay outlines an argument which shows that when BP is interpreted in his model for ZF + DC then it becomes a true statement. Hence BP is consistent with the axioms of ZF + DC provided Solovay's model exists. We adjoin BP as an axiom and denote the extended system by 'ZF + DC + BP'.

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In this paper certain propositions will be shown to be theorems of ZF + DC + BP. It is easy to show, by a Hamel base argument, that for each such proposition its negation is a theorem in ZFC (ZF with the axiom of choice adjoined). So these propositions can neither be proved nor disproved in ZF + DC, provided Solovay's model exists.

Let I be the axiom: *There exists an inaccessible cardinal.* Solovay uses the hypothesis that there exists a (transitive) model for ZFC + I when constructing his model.

From now onward we work in ZF + DC + BP. All our theorems are derived in this system.

LEMMA 1. *Let X and Y be separable metric spaces and let X be complete. Let $f: X \to Y$ be any function mapping X into Y. Then there exists a meagre set $N \subset X$ such that the restriction of f to $X \setminus N$ is continuous.*

Choose $\varepsilon > 0$. Let $\{y_r : r = 1, 2, \ldots\}$ be a countable dense subset of Y. For each r, let S_r be the open sphere centred on y_r with radius $\varepsilon/2$. Then $Y = \bigcup_1^\infty S_r$.

Let $A_1 = S_1$ and, for $n \geqq 1$, let $A_{n+1} = (\bigcup_1^{n+1} S_r) - (\bigcup_1^n S_r)$. So $Y = \bigcup_1^\infty A_n$, where each A_n is contained in an open sphere of radius $\varepsilon/2$ and $A_i \cap A_j = \varnothing$ for $i \neq j$.

Let $B_n = f^{-1}[A_n]$ for $n = 1, 2, \ldots$. Then $X = \bigcup_1^\infty B_n$ and $B_i \cap B_j = \varnothing$ for $i \neq j$.

For any n, B_n has the Baire property and so there is an open set U_n and a meagre set M_n, where $M_n = (B_n \setminus U_n) \cup (U_n \setminus B_n)$, such that $U_n \cap (X \setminus M_n) = B_n \cap (X \setminus M_n)$. Let M be the meagre set $\bigcup_1^\infty M_n$. Then $U_n \cap (X \setminus M) = B_n \cap (X \setminus M)$ for each n. Thus $B_n \cap (X \setminus M)$ is an open subset of $X \setminus M$ in the relative topology of $X \setminus M$.

Let J be the set of all natural numbers n for which $B_n \cap (X \setminus M)$ is not empty. By DC there exists a function ξ with domain J such that $\xi(n) \in B_n \cap (X \setminus M)$ for each n. Let h be the function defined on $X \setminus M$ by $h(x) = f(\xi(n))$ whenever $x \in B_n \cap (X \setminus M)$.

Let (z_j) $(j = 1, 2, \ldots)$ be any sequence in $X \setminus M$ which converges to a point z in $X \setminus M$. Then, for some $n \in J$, $B_n \cap (X \setminus M)$ is an open neighbourhood of z in the relative topology of $X \setminus M$. So there exists a natural number k such that $z_j \in B_n \cap (X \setminus M)$ whenever $j \geqq k$. Thus $h(z_j) = h(z)$ whenever $j \geqq k$. So $h:(X \setminus M) \to Y$ is continuous. Whenever $x \in X \setminus M$ then $x \in B_n \cap (X \setminus M)$ for some $n \in J$ and thus

$$d(h(x), f(x)) = d(f(\xi(n)), f(x)) < \varepsilon.$$

By putting $\varepsilon = 1/m$ $(m = 1, 2, \ldots)$ we can find a sequence of functions (h_m) $(m = 1, 2, \ldots)$ and a sequence of meagre sets (N_m) $(m = 1, 2, \ldots)$ such that h_m is a continuous map of $X \setminus N_m$ into Y and $d(h_m(x), f(x)) < 1/m$

for each $x \in X \setminus N_m$. Let N be the meagre set $\bigcup_1^\infty N_m$. Then (h_m) $(m = 1, 2, \ldots)$ converges uniformly to f on $X \setminus N$. So f is continuous on $X \setminus N$.

THEOREM 2. *Let X and Y be separable metrizable topological groups and let X be complete. Let $H: X \to Y$ be any group homomorphism. Then H is continuous.*

Let (x_n) $(n = 1, 2, \ldots)$ be a sequence in X converging to a point x. By Lemma 1, there is a meagre set M such that H is continuous when restricted to $X \setminus M$.

By the Baire category theorem, which is valid in ZF + DC, there exists $z \in X$ such that z is not in the meagre set $x^{-1}M \cup \bigcup_1^\infty (x_n^{-1}M)$. Thus $xz \in X \setminus M$ and $x_n z \in X \setminus M$ for each n. Hence $H(xz) = \lim H(x_n z)$. Since H is a homomorphism, $H(z) = \lim H(x_n)$.

The elegant argument used in Theorem 2 is due to Banach, see Theorem 4, Chapter 1 [1]. I wish to thank Professor A. Wilansky for drawing my attention to this reference.

In the following we do not require Fréchet spaces to be locally convex.

THEOREM 3. *Let X be any Fréchet space and let Y be a separable metrizable topological vector space. Let $T: X \to Y$ be a linear map. Then T is continuous.*

Let (x_n) $(n = 1, 2, \ldots)$ be any sequence in X which converges to zero. Let X_0 be the closed linear span of $\{x_n : n = 1, 2, \ldots\}$ so that X_0 is a separable Fréchet space. Then, by the preceding theorem, the restriction of T to X_0 is continuous. Thus $Tx_n \to 0$ as $n \to \infty$. So T is continuous.

COROLLARY 4. *Each linear functional on a Fréchet space is continuous.*

THEOREM 5. *Let X and Y be Fréchet spaces and let $T: X \to Y$ be a linear map. If there exist enough functionals on Y to separate the points of Y then T is continuous.*

Let (x_n) $(n = 1, 2, \ldots)$ be a sequence in X converging to x and suppose (Tx_n) $(n = 1, 2, \ldots)$ converges to y. For any functional ϕ on Y, ϕ is continuous on Y and ϕT is continuous on X. Thus

$$\phi(y) = \lim \phi(Tx_n) = \lim \phi T(x_n) = \phi T(x).$$

So $Tx = y$. It now follows by the closed graph theorem that T is continuous.

It must be emphasised that discontinuous linear operators, defined on incomplete spaces, arise naturally in ZF + DC. For example, there is an abundance of unbounded operators defined on dense subspaces of a Hilbert space. But, for linear operators defined on the *whole* of a Hilbert space the following theorem holds in ZF + DC + BP.

THEOREM 6. *Let H be a Hilbert space and let $T:H \to H$ be a linear operator defined on the whole of H. Then T is bounded.*

Let H be any Hilbert space. Then, for each nonzero x in H, the linear functional f, defined by $f(y) = \langle y, x \rangle$, does not vanish at x. So H has a separating family of linear functionals.

This implies that, in ZFC, we cannot obtain discontinuous operators on (the whole of) a Hilbert space except by invoking an 'uncountable' form of the axiom of choice.

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