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REPRESENTATIONS OF GENERALIZED MULTIPLIERS OF L^p-SPACES OF LOCALLY COMPACT GROUPS

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The objective of this note is to announce a class of generalized multipliers between L^p-spaces of locally compact groups and some characterizations obtained by the author which generalize the classical representations of Figà Talamanca, Gaudry, Rieffel and others. If G is a locally compact group, let $L^p(G)$, $1 \leqq p \leqq \infty$, denote the corresponding Lebesgue spaces relative to a fixed Haar measure dx (with the convention that dx is normalized if G is compact). Let L_x for $x \in G$ denote the left translation operator on $L^p(G)$ given by $L_x f(y) = f(x^{-1}y)$. Let G, H, and K be locally compact groups and let $\theta: K \to G$ and $\psi: K \to H$ be continuous group homomorphisms. Let $1 \leqq p, q \leqq \infty$. We define a *$(\theta, p; \psi, q)$-multiplier* to be a bounded linear transformation $T: L^p(G) \to L^q(H)$ such that $T \circ L_{\theta(z)} = L_{\psi(z)} \circ T$ for all $z \in K$. Let $\mathrm{Hom}_K(L^p(G), L^q(H))$ denote the Banach space with the operator norm of all $(\theta, p; \psi, q)$-multipliers of $L^p(G)$ into $L^q(H)$. When $G = H = K$ and $\theta = \psi = \mathrm{id}_G$ (the identity map on G) then a $(\mathrm{id}_G, p; \mathrm{id}_G, q)$-multiplier is a "classical" (p, q)-multiplier of $L^p(G)$ into $L^q(G)$.

In [**1**] and [**2**], Figà-Talamanca and Gaudry have shown the "classical" multiplier space $\mathrm{Hom}_G(L^p(G), L^q(G))$ is isometrically isomorphic to the Banach space dual of the Banach space $A_p^{q'}(G)$ [**14**, Definitions 3.2 and 5.4] of functions on G for LCA groups G where $1/q + 1/q' = 1$. Rieffel [**14**] has extended this representation to amenable locally compact groups (using an approximation theorem of C. S. Herz when G is possibly noncompact). The representation for general G is still an open problem.

In this note we describe extensions of the above cited representations to the space of $(\theta, p; \psi, q)$-multipliers. Our approach parallels that of Rieffel in [**14**] by using tensor products of Banach modules. We assume familiarity with the general results concerning tensor products of Banach modules in [**13**]; specifically, if V and W are left and right Banach A-modules for a Banach algebra A then by [**13**, Corollary 2.13]

$$(1.1) \qquad (V \otimes_A W)^* \cong \mathrm{Hom}_A(V, W^*),$$

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where W^* is considered as a left Banach A-module under the adjoint action induced by the right action of A on W and $\mathrm{Hom}_A(V, W^*)$ is the Banach space of A-module bounded linear transformations of V into W^*.

We proceed to define module actions of $L^1(K)$ on $L^p(G)$ and $L^q(H)$. For $1 \leqq p \leqq \infty$, regard $L^p(G)$ as a left Banach $L^1(K)$-module under the action $(f, g) \to f *_\theta g$ where

$$f *_\theta g(x) = \int_K f(z) g(\theta(z)^{-1} x)\, dz \qquad (x \in G),$$

and $\|f *_\theta g\|_p \leqq \|f\|_1 \|g\|_p$. For $1 \leqq q \leqq \infty$, regard $\bar{L}^q(H)$ ($= L^q(H)$) as the right Banach $L^1(K)$-module under the action $(f, h) \to f^{\sim} *_\psi h$ where $f^{\sim}(z) \equiv f(z^{-1})\, \Delta_K(z^{-1})$, $z \in K$. Note that $L^p(G)$ and $L^q(H)$ are also left and right K-modules [**13**, Definition 1.1(b)] under the actions $(z, g) \to L_{\theta(z)} g$ and $(z, h) \to L_{\psi(z)^{-1}} h$, respectively, and that when $1 \leqq p, q < \infty$, these actions are strongly continuous and uniformly bounded [**13**, Definition 1.1(d)], and the essential Banach $L^1(K)$-actions they induce [**13**, p. 447] are precisely the above described $L^1(K)$-actions on $L^p(G)$ and $\bar{L}^q(H)$. Finally, when $1 \leqq q < \infty$, note that the adjoint action of $f \in L^1(K)$ on $L^{q'}(H)$, under which $L^{q'}(H)$ becomes a left Banach $L^1(K)$-module, is ψ-convolution by f; a similar statement applies to the K-module actions.

Since the K-module and $L^1(K)$-module tensor products of $L^p(G)$ and $\bar{L}^q(H)$ are isomorphic for $1 \leqq p, q < \infty$ [**13**, Theorem 3.14], we have by relation (1.1) the isometric isomorphism

$$(L^p(G) \otimes_{L^1(K)} \bar{L}^q(H))^* \cong \mathrm{Hom}_K(L^p(G), L^{q'}(H))$$

for all $1 \leqq p, q < \infty$ and $1/q + 1/q' = 1$. Consequently, analogous to the classical case [**14**], it suffices to obtain a concrete representation of the tensor space $L^p(G) \otimes_{L^1(K)} \bar{L}^q(H)$.

Let Q denote the closed subgroup and closure in $G \times H$ of the subgroup $\{(\theta(z), \psi(z)) : z \in K\}$. Let $G \otimes_K H$ denote the locally compact homogeneous space, $(G \times H)/Q$, of left cosets of Q in $G \times H$. Equip $G \times H$ with the product Haar measure $dx \otimes dy$, and let $d(u, v)$ denote the Haar measure on Q. According to [**16**, Chapter 8, §§1 and 2] there is a positive quasi-invariant measure $d_{\boldsymbol{q}}(x, y)^{\boldsymbol{\cdot}}$ on $G \otimes_K H$ corresponding to a strictly positive continuous solution $\boldsymbol{q}(x, y)$ on $G \times H$ to the functional equation

$$\boldsymbol{q}(xu, yv) = \boldsymbol{q}(x, y)\, \Delta_Q(u, v)/\Delta_G(u)\, \Delta_H(v), \qquad (x, y) \in G \times H, \quad (u, v) \in Q$$

such that $\int_{G \times H} F\, dx \otimes dy = \int_{G \otimes_K H} T_{Q,\boldsymbol{q}} F\, d_{\boldsymbol{q}}(x, y)^{\boldsymbol{\cdot}}$ for all $F \in L^1(G \times H)$ where $T_{Q,\boldsymbol{q}}$ is the canonical map $L^1(G \times H) \to L^1(G \otimes_K H)$ given by

$$T_{Q,\boldsymbol{q}} F(x, y)^{\boldsymbol{\cdot}} = \int_Q \frac{F(xu, yv)}{\boldsymbol{q}(xu, yv)}\, d(u, v), \qquad (x, y)^{\boldsymbol{\cdot}} = (x, y)/Q.$$

If G and H are compact then one can take $q \equiv 1$ and $d(x, y)^\cdot = d_q(x, y)^\cdot$ is invariant [**16**, Chapter 8, §1.4]. Define $f \wedge g(x, y) = f(x)g(y)$ for functions f on G and g on H.

For reasons of generality we consider Beurling algebras on locally compact groups [**15**]. Let $L^1_\omega(G)$, $L^1_\eta(H)$, and $L^1_\zeta(K)$ denote the Beurling algebras on G, H, and K with respect to (upper semicontinuous) weight functions on G, H, and K, respectively (see [**15**]).

LEMMA 1. *$L^1_\omega(G)$ is a left Banach $L^1_\zeta(K)$-module under the action $(f, g) \to f *_\theta g$ if and only if there is an $M \geqq 0$ such that*

(i) $\omega(\theta(z)x) \leqq M\zeta(z)\omega(x)$ *for loc. a.e.* $(z, x) \in K \times G$.

$L^1_\eta(H)$ is a right Banach $L^1_\zeta(K)$-module under the action $(f, h) \to f^\sim *_\psi h$ if and only if there is an $M \geqq 0$ such that

(ii) $\eta(\psi(z)^{-1}y) \leqq M\zeta(z)\eta(y)$ *for loc. a.e.* $(z, y) \in K \times H$.

Before stating one of the main results let

$$\omega^* \otimes_\zeta \eta^*(x, y)^\cdot \equiv \inf_Q \omega((xu)^{-1})\eta((yv)^{-1})$$

for $(x, y)^\cdot = (x, y)/Q$. Then $\omega^* \otimes_\zeta \eta^*$ is a positive upper-semicontinuous function bounded away from zero on $G \otimes_K H$. Let $L^1_{\omega^* \otimes_\zeta \eta^*}(G \otimes_K H)$ be the Lebesgue space $L^1(G \otimes_K H, \omega^* \otimes_\zeta \eta^* \, d_q(x, y)^\cdot)$.

THEOREM 1. *Let ω, η, and ζ be weight functions on the locally compact groups G, H, and K, respectively, satisfying* (i) *and* (ii) *of Lemma* 1. *Then*

$$L^1_\omega(G) \otimes_{L^1_\zeta(K)} \bar{L}^1_\eta(H) \cong L^1_{\omega^* \otimes_\zeta \eta^*}(G \otimes_K H)$$

where the isomorphism is linear and isometric, and the element $g \otimes h$ corresponds to $T_{Q,q}(g^\sim \wedge h^\sim)$.

The isomorphism of Theorem 1 was proved (without the condition of isometry) for LCA groups G, H, and K, continuous open homomorphisms θ and ψ, and for constant one weight functions by Gelbaum [**4**] and Natzitz [**12**]. In [**10**] this author has characterized the tensor module $L^1(G) \otimes_{L^1(K)} L^1(H)$ for all LCA groups G, H, and K and arbitrary algebra actions of $L^1(K)$ on $L^1(G)$ and $L^1(H)$, respectively. Analogous representations for tensor products of commutative semigroup algebras have been obtained by Lardy [**11**] and for H^*-algebras by Grove [**6**].

A proof of Theorem 1 amounts to showing that the closed linear subspace J of $L^1(G) \otimes_\gamma L^1(H)$ whose quotient with $L^1(G) \otimes_\gamma L^1(H)$ defines $L^1(G) \otimes_{L^1(K)} \bar{L}^1(H)$ [**13**, §2.2] corresponds under the Grothendieck [**5**, p. 90] and Johnson [**9**] (cf. [**3**, Remark 3, p. 304]) isomorphism $L^1(G) \otimes_\gamma L^1(H) \cong L^1(G \times H)$ to the closed linear subspace $J^1(G \times H, Q)$ of $L^1(G \times H)$, whose quotient with $L^1(G \times H)$ is isomorphic to $L^1(G \otimes_K H)$ (cf., [**16**, Chapter 8, §2.3(6)]; this establishes the isomorphism for the case in which the weight functions are constantly one. The general case is handled in a similar fashion but requires an extension of the isomorphism $L^1_\omega(G/H) \cong L^1_\omega(G)/J^1_\omega(G, H)$ of Reiter [**16**, Chapter 3, §7.4] for closed normal subgroups H of G to admit arbitrary closed subgroups H of G.

LEMMA 2. *If G, H, p, and q satisfy one of the following three conditions*:
(i) *G and H are compact and $1 \leqq p, q < \infty$*,
(ii) *G is compact, $1 \leqq p < \infty$, and $q = 1$*,
(iii) *H is compact, $p = 1$, and $1 \leqq q < \infty$*,
then $T_{Q,q}(g^\sim \wedge h^\sim) \in L^r(G \otimes_K H)$ (where $r = \min(p, q)$) for all $g \in L^p(G)$ and $h \in L^q(H)$,

$$\|T_{Q,q}(g^\sim \wedge h^\sim)\|_r \leqq \|g\|_p \|h\|_q,$$

and $(g, h) \to T_{Q,q}(g^\sim \wedge h^\sim)$ is an $L^1(K)$-balanced (bounded) bilinear map.

DEFINITION 1. With the hypotheses (i), (ii) or (iii) of Lemma 2, let $\mathscr{A}^q_p(G \otimes_K H)$ denote the space of all $F \in L^r(G \otimes_K H)$ which have at least one expansion of the form $F = \sum_1^\infty T_{Q,q}(g_n^\sim \wedge h_n^\sim)$ where $(g_n) \subseteqq L^p(G)$, $(h_n) \subseteqq L^q(H)$, and $\sum_1^\infty \|g_n\|_p \|h_n\|_q < \infty$ (with the expansion for F converging in the norm of $L^r(G \otimes_K H)$). If $\mathscr{A}^q_p(G \otimes_K H)$ is equipped with the norm

$$F \to \|F\| = \inf\left\{\sum_1^\infty \|g_n\|_p \|h_n\|_q : F = \sum_1^\infty T_{Q,q}(g_n^\sim \wedge h_n^\sim)\right\},$$

then it is a Banach space.

The second of our main results is

THEOREM 2. *If G, H, p, and q satisfy one of the conditions* (i), (ii) or (iii) *of Lemma* 2 *then*

$$L^p(G) \otimes_{L^1(K)} \bar{L}^q(H) \cong \mathscr{A}^q_p(G \otimes_K H)$$

where the isomorphism is algebraic and isometric and the element $g \otimes h$ corresponds to $T_{Q,q}(g^\sim \wedge h^\sim)$.

The proof of Theorem 2 is based on Theorem 1 and a lemma concerning the approximation of $(\theta, p; \psi, q)$-multipliers by $(\theta, 1; \psi, \infty)$-multipliers. We show to every $(\theta, p; \psi, q)$-multiplier T for $1 \leqq p < \infty$, $1 \leqq q \leqq \infty$, there is a net (T_λ) of $(\theta, 1; \psi, \infty)$-multipliers such that the restriction of T_λ to the space $\mathscr{K}(G)$ of continuous functions on G with compact support has a (unique) bounded linear extension to a $(\theta, p; \psi, q)$-multiplier S_λ and the (S_λ) converge ultra-weakly to T. This approximation lemma is used to show that the canonical map

$$L^p(G) \otimes_{L^1(K)} \bar{L}^q(H) \ni \sum_1^\infty g_n \otimes h_n \mapsto \sum_1^\infty T_{Q,q}(g_n^\sim \wedge h_n^\sim) \in \mathscr{A}^q_p(G \otimes_K H)$$

has trivial kernel.

Consider the classical case when $G = H = K$ and $\theta = \psi = \mathrm{id}_G$. Set $\boldsymbol{q}(x, y) = \Delta_G(y^{-1})$ and note that $Q \cong G$ and $\tau((x, y)/Q) = xy^{-1}$ is a topological isomorphism of $G \otimes_G G$ onto G. In this case it can be easily shown that

$$T_{Q,\boldsymbol{q}}(g^{\sim} \wedge h^{\sim})(\cdot) = g^{\sim} * h(\tau(\cdot)), \qquad g, h \in \mathscr{K}(G).$$

Thus (for compact G) it is seen that the adjoint of τ, τ^*, induces an isometric isomorphism of the space $\mathscr{A}_p^q(G \otimes_G G)$ with the space $A_p^q(G)$ [**14**, Definition 3.2]. As an application of Theorem 2 we have

$$L^1(\boldsymbol{R}) \otimes_{l^1(\boldsymbol{Z})} L^q(\Delta_{\boldsymbol{a}}) \cong \mathscr{A}_1^q(\Sigma_{\boldsymbol{a}}), \qquad (1 \leqq q < \infty),$$

where $\boldsymbol{R}$ = reals, $\boldsymbol{Z}$ = integers, $\Delta_{\boldsymbol{a}}$ = $\boldsymbol{a}$-adic integers [**7**, (10.2)], $\Sigma_{\boldsymbol{a}}$ = $\boldsymbol{a}$-adic solenoid [**7**, (10.12)], and where θ and ψ are the natural inclusions of $\boldsymbol{Z}$ into $\boldsymbol{R}$ and $\Delta_{\boldsymbol{a}}$, respectively.

Our third main result is an extension of the classical result of Hörmander [**8**, Theorem 1.1] which asserts that $\mathrm{Hom}_G(L^p(G), L^q(G)) = \{0\}$ if G is noncompact and $1 \leqq q < p < \infty$. We require first

DEFINITION 2. K is said to be (θ, ψ)-*compact* if there is a subset A in K such that $\theta(A)$ and $\psi(K \sim A)$ are precompact in G and H, respectively.

THEOREM 3. *Let $1 \leqq q < p < \infty$. If K is (θ, ψ)-noncompact, then* $\mathrm{Hom}_K(L^p(G), L^q(H)) = \{0\}$.

COROLLARY 1. *If K is (θ, ψ)-noncompact, $1 < p$, $q < \infty$, and $1/p + 1/q < 1$, then* $L^p(G) \otimes_{L^1(K)} \bar{L}^q(H) = \{0\}$.

The proof of Theorem 3 is based on the equivalence of (θ, ψ)-noncompactness with the property that to each pair of compact subsets U in G and V in H, there is a $z \in K$ such that $(\theta(z)U) \cap U = \varnothing$ and $(\psi(z)V) \cap V = \varnothing$. At this point the Hörmander method of "shifting" applies.

In another paper, this author and W. D. Pepe consider the problem of characterizing these generalized multipliers when the range space is $L^1(H)$ or $M(H)$, and thereby obtain generalizations of Wendel's theorem. The results are similar to those obtained above.

Detailed proofs of the above results will appear elsewhere.

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