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### SOME RESULTS IN E. CARTAN'S THEORY OF ISOPARAMETRIC FAMILIES OF HYPERSURFACES<sup>1</sup>

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In a series of papers [1]–[4], E. Cartan developed the theory of isoparametric families of hypersurfaces and proposed a number of problems. The purpose of the present note is to announce the following three results. First, we construct a series of isoparametric families of hypersurfaces  $M_t^{2n}$  in  $S^{2n+1}$ ,  $n \geq 2$ , thus providing an affirmative answer to one of Cartan's problems. Second, we show that each focal variety belonging to an isoparametric family  $M_t^n$  in  $S^{n+1}$  admits a global submanifold structure if  $M_t^n$  consists of compact hypersurfaces. Finally, we prove that each focal variety of  $M_t^n$  is a minimal submanifold in  $S^{n+1}$ .

In §1, we recall, very briefly, some basic facts from Cartan's work. In §2, we give the construction of isoparametric families  $M_t^{2n}$  in  $S^{2n+1}$ . In §3 we discuss focal varieties, in particular, from a global point of view. In §4 we deal with minimality of focal varieties. The details will appear elsewhere together with a systematic account of Cartan's theory of isoparametric families.

**1. Isoparametric family of hypersurfaces.** A connected hypersurface  $M^n$  in the sphere  $S^{n+1}$  (unit hypersphere in Euclidean space  $R^{n+2}$ ) is said to have constant principal curvatures if there are distinct constants  $a_1, \dots, a_p$  which, for a suitable choice of a unit normal vector field  $\xi$ , represent all the distinct principal curvatures at every point. In this case, the multiplicity  $v_i$  of each  $a_i$  remains the same throughout  $M$ . Of course,  $n = \sum_{i=1}^p v_i$ . Let  $M_t^n$  be a family of parallel hypersurfaces obtained by moving each point of  $M^n$  by distance  $t$  along the geodesic in the direction

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of  $\xi$ . Then, for each  $t$ ,  $M_t^n$  has constant principal curvatures  $\cot(\theta_i - t)$ ,  $1 \leq i \leq p$ , where  $\theta_i$  is the angle between  $-\pi$  and  $\pi$  such that  $\cot \theta_i = a_i$ ,  $1 \leq i \leq p$ . The family  $M_t^n$  is called an isoparametric family of hypersurfaces. We content ourselves here with local considerations. If  $M^n$  is compact, then there is some  $c > 0$  such that, for each  $t$ ,  $|t| < c$ ,  $M_t^n$  is a nonsingular hypersurface of  $S^{n+1}$ . Even locally, however, how big  $t$  can be is a question of focal point which we discuss in §3.

For a hypersurface  $M^n$  with distinct constant principal curvatures  $a_1, \dots, a_p$  with multiplicities  $v_1, \dots, v_p$  in  $S^{n+1}$  there is a basic identity proved by Cartan:

$$(*) \text{ For each } i, \sum_{k \neq i} v_k(1 + a_i a_k)/(a_i - a_k) = 0.$$

Cartan notes that it is of interest to discover a geometric proof of this identity. We shall later find a geometric interpretation of (\*).

One of the problems posed by Cartan is: Does there exist a hypersurface  $M^n$  in  $S^{n+1}$  with more than three distinct constant principal curvatures, not all of the same multiplicity? We answer this question affirmatively in §2.

In Cartan's theory, isoparametric families of hypersurfaces are closely related to families of level hypersurfaces for a certain class of functions on  $S^{n+1}$ . For a differentiable function  $F$  on  $S^{n+1}$  (or on an open subset of  $S^{n+1}$ ), let

$$\Delta_1 F = \|dF\|^2 \quad \text{and} \quad \Delta_2 F = \text{Laplacian of } F,$$

and consider the following condition:

$$(**) \Delta_1 F \text{ and } \Delta_2 F \text{ are functions of } F.$$

Under this condition, the level hypersurface

$$M_c^n = \{x \in S^{n+1}; F(x) = c, \Delta_1 F(x) \neq 0\}$$

is a (nonsingular) hypersurface with constant principal curvatures, if it is nonempty. By assuming  $\Delta_1 F = 1$ , the family of level hypersurfaces  $M_t^n = \{x \in S^{n+1}; F(x) = t\}$  constitutes an isoparametric family for a suitable range of  $t$ . Cartan showed that if  $M^n$  has  $p$  distinct principal curvatures with the same multiplicity  $v$  (so that  $n = pv$ ), then it can be obtained as the level hypersurface  $F = \cos pt$ , where  $F$  is a harmonic homogeneous polynomial of degree  $p$  (restricted to  $S^{n+1}$ ).

2. **A series of isoparametric families  $M_t^{2n}$  in  $S^{2n+1}$ .** Let  $R^{n+1}$  be the standard  $(n + 1)$ -dimensional real vector space with the usual inner product  $\langle x, y \rangle$ . If we regard the  $(n + 1)$ -dimensional complex vector space  $C^{n+1}$  as a real vector space, then we have  $C^{n+1} = R^{n+1} + iR^{n+1}$ . By writing  $z, w \in C^{n+1}$  in the form  $z = x + iy, w = u + iv$ , the real inner product in  $C^{n+1}$  is given by

$$\langle x + iy, u + iv \rangle = \langle x, u \rangle + \langle y, v \rangle.$$

Let  $S^{2n+1} = \{z \in C^{n+1}; \|z\| = 1\}$  and consider a function

$$F(z) = (\|x\|^2 - \|y\|^2)^2 + 4\langle x, y \rangle^2 \quad \text{for } z = x + iy$$

on  $S^{2n+1}$ . The function  $F$  satisfies condition (\*\*):

$$\Delta_1 F = 16F(1 - F) \quad \text{and} \quad \Delta^2 F = 16(1 - 2F).$$

For each  $t, 0 < t < \pi/4$ , let  $M_t^{2n} = \{z \in S^{2n+1}; F(z) = \cos^2 2t\}$ .

We have then

**THEOREM 1.**  $M_t^{2n}, 0 < t < \pi/4$ , forms an isoparametric family of hypersurfaces in  $S^{2n+1}$ . For a fixed  $t$ , the hypersurface  $M_t^{2n}$  has the following principal curvatures:

$$(1 + \sin 2t)/\cos 2t, \quad (-1 + \sin 2t)/\cos 2t \quad \text{of multiplicity } 1;$$

$$\tan t, \quad -\cot t \quad \text{of multiplicity } n - 1.$$

Moreover,  $M_t^{2n}$  is the image of the imbedding of  $S^1 \times S_{n+1,2} \rightarrow S^{2n+1}$ , where  $S_{n+1,2}$  is the Stiefel manifold of all orthonormal pairs of vectors  $x, y \in R^{n+1}$ , given by

$$(e^{i\theta}, (x, y)) \rightarrow e^{i\theta/2}(\cos tx + i \sin ty).$$

**REMARK 1.** For  $n = 2$ , the isoparametric family  $M_t^4$  in  $S^5$  appears as one of the examples studied by Cartan. For  $n \geq 3$ ,  $M_t^{2n}$  gives an affirmative answer to the problem quoted in §1.

**REMARK 2.** R. Takagi and T. Takahashi [6] answered the same question by determining homogeneous hypersurfaces in  $S^n$  (which, obviously, have constant principal curvatures). Our hypersurface in Theorem 1 corresponds to the fifth example on their Table II, for which they have not given the actual values of principal curvatures. On the other hand, they have found other examples which answer the question of Cartan.

**3. Focal varieties.** For a hypersurface  $M^n$  in  $S^{n+1}$ , the notion of focal point can be defined as follows. Let  $\xi$  be a field of unit normal vectors. A point  $y_0$  in  $S^{n+1}$  is a focal point of  $(M, x_0)$ , where  $x_0 \in M$ , if  $y_0 = \cos t_0 x_0 + \sin t_0 \xi_{x_0}$  for some  $t_0$  and if the differential of the mapping  $(x, t) \rightarrow y = \cos tx + \sin t \xi_x$  is singular at  $(x_0, t_0)$ . A point  $y_0$  is called a focal point of  $M$  if it is a focal point of  $(M, x_0)$  for some  $x_0 \in M$ .

It is easily seen that  $y_0 = \cos t_0 x_0 + \sin t_0 \xi_{x_0}$  is a focal point of  $(M^n, x_0)$  if and only if  $\cot t_0$  is one of the principal curvatures (eigenvalues of the second fundamental form  $A$  corresponding to  $\xi$ ).

Now assume that  $M^n$  has distinct constant principal curvatures  $a_1, \dots, a_p$  of multiplicities  $v_1, \dots, v_p$ . If we let

$$T_i(x) = \{X \in T_x(M^n); AX = a_i X\}, \quad x \in M^n,$$

then we obtain distributions  $T_1, \dots, T_p$  of dimensions  $v_1, \dots, v_p$  on  $M$ . It can be shown easily that each distribution is integrable and the maximal integral manifold  $M_i(x)$  of  $T_i$  through  $x$  is totally geodesic in  $M^n$  and umbilical as a submanifold of  $S^{n+1}$ . (Thus, if  $M^n$  is connected and compact,  $M_i(x)$  is a  $v_i$ -dimensional small sphere of  $S^{n+1}$ .)

For each  $i$ ,  $1 \leq i \leq p$ , we define a differentiable mapping  $f_i$  of  $M^n$  into  $S^{n+1}$  by  $f_i(x) = \cos \theta_i x + \sin \theta_i \xi_x$ , where  $\cot \theta_i = a_i$  as before. The differential of  $f_i$  at  $x$  has  $T_i(x)$  as the null space and is injective on the sum  $\sum_{k \neq i} T_k(x)$ . This implies that for each point  $x$  there is a local coordinate system  $\{u^1, \dots, u^n\}$  with origin  $x$  such that  $f_i$  is a one-to-one immersion of the slice  $u^1 = \dots = u^{v_i} = 0$  into  $S^{n+1}$ , the image being in  $f_i(M^n)$ . Thus we may consider  $f_i(M^n)$  as a submanifold of dimension  $n - v_i$  in a neighborhood of  $f_i(x)$ . We call  $f_i(M^n)$  a *focal variety* of  $M^n$  for each  $i$ . If  $M^n$  is compact, we can indeed give a global submanifold structure to each focal variety.

**THEOREM 2.** *Let  $M^n$  be a connected compact hypersurface in  $S^{n+1}$  with constant principal curvatures. Each focal variety  $f_i(M^n)$  is a submanifold of  $S^{n+1}$  in the following sense. There exists an  $(n - v_i)$ -dimensional differentiable manifold  $V_i$ , a differentiable mapping  $\pi_i$  of  $M$  onto  $V_i$ , and a differentiable imbedding  $g_i$  of  $V_i$  into  $S^{n+1}$  such that  $f_i = g_i \circ \pi_i$ , in particular,  $f_i(M) = g_i(V_i)$ .*

For the proof, we make use of the result of Palais [5] concerning the space of leaves of a regular foliation.

#### 4. Minimality of focal varieties. We have

**THEOREM 3.** *Let  $M^n$  be as in Theorem 2. Each focal variety is a minimal submanifold of  $S^{n+1}$ .*

The question of minimality seems to have escaped the attention of Cartan, although examples of focal varieties which appear in his papers include such notable minimal submanifolds as the Veronese surface in  $S^4$ .

Theorem 3 can be stated as a local result without assuming compactness of  $M^n$ . Since minimality is a local condition, the proof is carried out by local computations of the traces of the second fundamental forms of  $f_i(M^n)$ . The tangent space to  $f_i(M^n)$  at  $f_i(x)$  can be identified with  $\sum_{k \neq i} T_k(x)$  through Euclidean parallelism in  $R^{n+2}$ . The normal space to  $f_i(M^n)$  at  $f_i(x)$  is given as follows. First, the tangent vector  $\eta$  of the geodesic  $\cos tx + \sin t \xi_x$  at  $t = \theta_i$ , namely,  $\eta = -\sin \theta_i x + \cos \theta_i \xi_x$ , is in the normal space. Second, every  $Z \in T_i(x)$ , can be considered (through parallel translation in  $R^{n+2}$ ) as a normal vector to  $f_i(M^n)$ . The vectors  $\eta$  and  $Z$ ,  $Z \in T_i(x)$ , span the normal space.

It is not difficult to show that the trace of the second fundamental form of  $f_i(M^n)$  corresponding to every  $Z$  is 0. On the other hand, computation shows that the trace of the second fundamental form of  $f_i(M^n)$  corresponding to  $\eta$  turns out to be precisely the sum

$$\sum_{k \neq i} v_k \frac{1 + a_i a_k}{a_i - a_k}$$

in the basic identity (\*) of Cartan. We have thus a proof of Theorem 3 as well as a geometric interpretation of (\*).

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