Lie Algebras and Quantum Mechanics and *Vector Bundles in Mathematical Physics* and their relationship to the other works of Robert Hermann.

We shall use the following code, to be able to refer to Professor Hermann's books conveniently.

LG	*Lie groups for physicists*, Benjamin, N.Y., 1966.
DG	*Differential geometry and the calculus of variations*, Academic Press, N.Y., 1968.
LA	*Lie algebras and quantum mechanics*, Benjamin, N.Y., 1969.
FA	*Fourier analysis on groups and partial wave analysis*, Benjamin, N.Y., 1969.
VB-1, VB-2	*Vector bundles in mathematical physics*. I, II, Benjamin, N.Y., 1970.
MP-1, MP-2	*Lectures on mathematical physics*. I, II, Benjamin, N.Y., 1970.
PS	*Geometry, physics and systems*, Dekker, N.Y., 1973
PA	*Some physical aspects of Lie group theory*, mimeographed, Université de Montréal, 1972.

TABLE 1.

(In addition, Professor Hermann has announced plans for a series of volumes entitled *Interdisciplinary mathematics* which continue his program.)

Before we comment on these works, and on LA and VB in particular, it might be helpful to list some of the *major* topics found in these books. We do this since topics are often repeated in various contexts and degrees of development.

Calculus of variations	DG(part 2), VB-1(Ch. 2), LA(Ch. V), PS(Ch. III, IV)
Classical limit of quantum mechanics	VB-2(Ch. V, VII)
Classical mechanics (geometry of symplectic structures, etc.)	DG(Ch. 33), MP-1(Ch. II, IV), LA(Ch. II), VB-1(Ch. V), VB-2 (Ch. II), PS(Ch. II, III)
Constrained systems	DG(Ch. 12, 33), LA(Ch. II)
Current algebras	LA(Ch. III, IV)
Differential geometry (basic theory)	DG(part III)

Differential operators (symbols, invariance properties, Dirac equation, etc.)	LG(Ch. 10), VB-1(Ch. VII, VIII), VB-2(Ch. VI, IX)
Elementary particles	LG(Ch. 17, 18)
Fourier transform (generalizations to groups, etc.)	LG(Ch. 13), FA(Ch. II), PA(Ch. VII)
Functional analysis and related topics (basic theory)	MP-1(Ch. I, III, IV), MP-2 (Appendix), PA(Ch. III, IV, VIII, IX)
Generalized functions	MP-1(Ch. V), MP-2(Ch. II), LA (Ch. IX), PS(Ch. IV), PA(Ch. II)
Induced representations	LG(Ch. 9, 12), VB-1(Ch. VI)
Lagrangian systems (Noether's theorem, etc.)	VB-1(Ch. III, IV, VII), PS(Ch. III, IV)
Lie algebras (basic theory)	LG(Ch. 3–5), LA(Ch. I), VB-2(Ch. III)
Lie algebras (advanced topics)	LG(Ch. 5–8)
Lie groups (basic theory)	LG(Ch. 1–3), MP-2(Ch. 3 = pp. 131–517!), PA(Ch. I)
Lie groups (advanced topics)	LG(Ch. 11, 15), PA(Ch. X)
Manifold theory (calculus on manifolds, etc.)	DG(Part I), MP-1(Ch. II), VB-1 (Ch. I), VB-2(Ch. I), PS(Ch. I)
Path integrals	MP-2(Ch. IV)
Quantization	LA(Ch. II, VI), VB-2(Ch. IV, VII)
Quantum field theory	LA(Ch. V, VI, VII), MP-2(Ch. VIII), VB-2(Ch. V, VII, VIII, IX)
Scattering theory	LA(Ch. VII, IX), MP-2(Ch. V), PA(Ch. XII)
Symmetry groups in classical and quantum mechanics	DG(Ch. 16), LG(Ch. 16), VB-1 (Ch. II), VB-2(Ch. VIII), MP-1 (Ch. VI), PS(Ch. IV, V)
Statistical mechanics	MP-2(Ch. VII)
Thermodynamics	PS(Ch. VI)

TABLE 2.

The above table makes it clear that there are many interconnections between the various volumes. In fact, the volumes follow no definite logical sequence. However, we note that DG might be read first to get into the geometrical parts of the work, while MP-2(Ch. III), followed by parts of LG, gives background for Lie group theory; and VB-2(Ch. III), followed by parts of LG, gives adequate background in Lie algebra theory.

The most obvious characteristic of Professor Hermann's project is its vast scope. He has attempted to expound a large amount of material in the wilderness between mathematics and physics, and to give each side some idea of what is happening in the other. Being a mathematician, it is inevitable that he should emphasize the mathematical structures and problems underlying physical theories rather than the physics itself.

In describing his goals, we should distinguish his pre-1969 books (DG and LG) from his recent works. The earlier books, especially DG, are systematic, well-referenced, and scholarly treatises. His later books are intentionally informal, in the style of lecture notes. Professor Hermann has explained his objectives as follows:

> . . . I want to do for mathematical physics something like the job the Seminaire Cartan and Sophus Lie notes did for topology and Lie group theory back in the 50's—introduce new methods and ways of looking at old problems. Plus, get back to an older tradition of describing and developing ideas rather than just proving theorems, which I admit I find dull. [personal communication]

> Since my goal is to integrate a large chunk of contemporary mathematics and physics, I will again . . . adopt an informal and loose expository style. (In my "defense," it may be pointed out that this was in fact the style of sixty years ago—in the work of such men as Poincaré and E. Cartan—when mathematics and physics were much more closely integrated.) In the short run, this style requires more effort, imagination, and scholarly skills from the reader than the now current "theorem-proof" style, but I, for one, have found it to be more effective in the long term in achieving this integration. At least one can say that the theorem-proof style—as well as being more boring to write and to read—often bogs down in ponderous pedanticism.
>
> Much of this material has almost infinite ramifications and branchings in contemporary mathematics and physics. I will

then occasionally mention key names in a casual way, expecting, as part of the "scholarly skills" mentioned above, that the interested reader will refer to the index in *Mathematical Reviews* for further references. [from VB-2]

In this approach, "proofs" are not important for their own sake, but only as guideposts into new territory. In general, I think it is useful to keep in mind Poincaré's division of mathematicians into two categories, the "geometers" and "analysts". Now, this does not mean people who study geometry and analysis; but refers instead to a division between those who prefer one type of reasoning to another; say, those who like broad, qualitative, intuitive, i.e. geometry-like arguments, and those who treasure precise completely logical, deductive sorts of mathematics. Presumably, this arose in Poincaré's mind to explain his disagreements in matters of taste between his contemporaries like Cantor and Weierstrass, who were engaged in the process of driving out from the foundations of mathematics all the sloppiness that Poincaré found so stimulating.

Today, the "analysts" hold power. Most mathematicians recognize and practice only work that produces hardheaded, precise theorems that are "deep" and technically difficult, and that contribute to the solution of a problem that is recognized—by a consensus of mathematicians—as important and difficult. (If it is a "famous", old problem so much the better.) It is in the nature of things that a "geometer" will not do this, at least as his main activity. It seems to me, however, that if mathematics is ever to interact again with the outside world—say, with theoretical physics, at least—we must revive the "geometry" spirit, working in, of course, the advances in technique and formalism that have been provided by the labors of the "analysts". This book is written in that spirit. [from VB-1]

I believe that the geometry of manifolds can play a . . . unifying role in the application of mathematics to diverse areas of physics and engineering, and can even have a creative influence on research in the less traditional areas of applied mathematics, such as mathematical biology, economics and computer technology. My aim in this series of treatises is to develop this point of view.

Of course, there are many obstacles to a completely satisfactory realization of this goal. I know something about the

> geometry of manifolds, but cannot claim any expertise in the areas of application. All I can state is that I have spent a considerable portion of the last fifteen years studying various areas outside of pure mathematics from the point of view of this sort of unification, and that I am now ready to begin the work that might ultimately lead to a realization of this goal. Ideally, there would be a group or "school" working on this—analogous, say, to the Bourbaki effort in France at pure mathematics from a unified point of view,—but, due to the primitive intellectual state of "applied mathematics", in comparison with "pure", I have had to work alone. [from PS].

Certainly these books have many fine attributes. They contain many stimulating ideas, ranging from possible topics for student papers to lines of future research. Rapidity of publication and informality of style have facilitated the presentation of ideas and problems in their nascent state.

However, anyone who employs such an informal style is going to encounter certain inevitable criticisms, and Professor Hermann himself has anticipated this. Because the books have been published so rapidly, they lack indexes, contain numerous misprints and other slips, and are often not well organized. It is clear that Hermann has written on many topics before completely deciding what he wants to say, with the result that he returns several times to a given point, even within the same book. References and citations of the literature are not adequate, and many passages may be confusing to the nonexpert. Especially noticeable is his frequent use of quotation marks to encapsulate nondefined and nonreferenced terms. His style is well illustrated by the following rather typical extract from VB-1 (Ch. IX).

> . . . A complex-linear vector space V, with a Hermitian symmetric form $\langle v_1 \mid v_2 \rangle$ may be considered as a system of "states" for a quantum mechanical system. Thus, if $\langle v_1 \mid v_2 \rangle = re^{i\theta}$ with r, θ real numbers, then r represents some idea of "probability" for a "transition" between v_1 and v_2 while θ is some sort of "phase". (Of course, if the form is not positive definite, this is not quite standard—but perhaps such a generalization can be made.) Then, a real-linear transformation $A: V \to V$ such that
>
> $$|\langle Av_1 \mid Av_2 \rangle| = |\langle v_1 \mid v_2 \rangle| \qquad \text{for } v_1, v_2 \in V$$
>
> preserves "probabilities", hence can play the role of a physical "symmetry". This general mathematical picture of what a

"symmetry" should mean has been emphasized by Wigner [1].

Here the meaning of the words "states", "transition", etc. will certainly not be clear to a reader unfamiliar with quantum mechanics.

Hermann readily admits that his style calls on the scholarly skills of the reader to look up the references and terms. But this can be frustrating and time consuming. Moreover, researchers should note that these books contain a number of conjectures about ideas which are in fact known or wrong, and sometimes are treated better in another volume (see Table 2). This can be demoralizing, despite the fact that the books contain a wealth of good ideas.

Given human limitations, one must usually choose between a rough sketch of ideas now or a polished work ten years hence. Each has advantages and disadvantages. Physicists seems to prefer the first course, while mathematicians generally follow the second. Professor Hermann has chosen the advantages of quick publication. Even from the mathematical community's point of view this is not *a priori* a bad choice.

Before making any additional general comments, let us examine LA in some detail.

Lie algebras and quantum mechanics. Chapter I is entitled "The Lie algebra approach to classical and quantum mechanics".

Following Dirac, Hermann outlines in this chapter the familiar analogies between the classical Poisson bracket formalism and the quantum operator bracket formalism. A short section (§3) on states and observables includes some remarks on the probabilistic interpretation of quantum mechanics. To illustrate the sort of error that would have been caught if the manuscript had been carefully proofread, we note that Theorem 3.1 is false (and actually contradicts von Neumann's celebrated result on the nonexistence of "hidden variables"—see J. von Neumann, *Mathematical foundations of quantum mechanics*, Princeton Univ. Press, 1955); the error stems from a miscopying of indices in the midst of a calculation. The chapter contains some interesting remarks on the problem of "quantizing" classical systems. (Similar remarks are scattered throughout the book, and, in particular, almost the same material appears in Chapter VIII.) The last section is entitled "Classical and quantum systems with constraints"; however, nothing is said about quantum systems here (but see Chapter II).

Chapter II is called "Quantization of constrained systems". The title of this chapter is rather misleading. Most of the material deals with the construction of Poisson bracket operations for *classical* constrained systems, especially arising from Lagrange variational problems with constraints. The problem of quantizing these systems, i.e. representing the classical Lie algebras by operators on Hilbert space, is only mentioned. However, the chapter ends with a concrete example (particle restricted to a surface) which illustrates the general ideas of quantization. But this example is rather anticlimactic, first because it is quite standard and well known; second, because here the constraints involve the position coordinates only, and not the quite general constraints considered earlier by the author.

Chapters III and IV ("Current algebras and gauge groups; representation of the algebra of currents, and algebraic construction of Schwinger terms"). Here Hermann defines current algebras mathematically, and studies their representations by means of Lie algebra cohomology theory. It might be mentioned that these chapters have no connection at all with Chapters I and II; rather they seem to be a further study of ideas which the author presented in a series of papers on *Analytic continuation of group representations* (Comm. Math. Phys. 1966–1967). These chapters probably should be read in conjunction with the latter papers. Finally, we warn the reader that he will have to look elsewhere if he wishes any insight into the physical significance (or even the definition!) of "Schwinger terms".

Chapters V and VI ("A differential geometric formalism for multiple integral variational problems; applications to quantum field theory" and "Quantization of classical field theories using differential forms"). These chapters continue the discussion started in Chapters I and II. Chapter V presents a general geometric approach to the calculus of variations, following the ideas of E. Cartan (and treated elsewhere by Hermann—see VB-1 and PS). The main results of Chapter V are a very general construction of *classical* Poisson brackets, and a presentation of Noether's theorem on conserved currents and symmetries of a Lagrangian. There are merely allusions to *quantum* field theory; certainly no one not already familiar with the notion of a quantum field will learn about it from this chapter. The following is fairly typical: after a rather long calculation a formula is derived, concerning which the author says: "This relation may be considered as the differential geometric version of the 'Goldberger-Treiman' relation . . . ". The mathematical reader may be forgiven if he is tempted to shrug his shoulders at this point. He *may* be stimulated enough to try to find out what the "Goldberger-Treiman" relation is by wading through the physics literature. The ideas seem sound and quite interesting; it is unfortunate that one cannot help but feel frustrated by the style.

Chapter VI continues with *classical* field theory, despite the title.

It would be less of a misnomer to refer to "pre-quantization" here; the classical Poisson structures are discussed, but there is not a Hilbert space in sight. This chapter utilizes the paper of the author: *E. Cartan's*, *Geometric theory of partial differential equations*, Advances in Math. **1** (1965), 265–317. The chapter overlaps significantly with Chapter VIII, "A geometric viewpoint in quantum field theory" and also with VB-2, Chapter VII.

Chapters VII and IX ("Propagators and scattering" and "Scattering theory in generalized function spaces and on manifolds"). These chapters have essentially nothing to do with Lie algebras and quantum mechanics; rather, they present a more or less orthodox account of the ideas of scattering theory, both classical and quantum. The connection with the rest of the book is rather loose; the other chapters deal with problems of "kinematics", while these chapters deal with "dynamics", i.e. properties of solutions of the equations of motion. Scattering theory is treated elsewhere by Hermann, in particular, in MP-2. It seems that the latter presentation is better organized and clearer, although less ambitious. (When Hermann focuses on a smaller number of ideas he attains much greater clarity.) There is much duplication in Chapters VII and IX, and they should have been combined.

In connection with the duplication, both within this book, and with his other books, a notion comes to mind that Hermann's books are written using the methods of holography: thus each small chunk embodies a blurry version of the entire work.

The most striking defect of LA (and much of Hermann's other work) is the absence of any serious attempt at a physical interpretation of the mathematics presented. Physical jargon is dragged in, but rarely explained. Probably the worst instance is the introduction of "current algebras" with *no* physical examples, and no motivation for their study beyond the casual remark that Gell-Mann is interested in the subject. (Admittedly, this *is* a weighty reason, but hardly an intrinsic one.) It is no disgrace to fall short of the Olympian heights of von Neumann and Weyl, but the attempt should be made. Since Weyl's book on quantum mechanics was published within two or three years of the discoveries of Heisenberg and Schrodinger, we have at least *one* important example which shows that timeliness is not necessarily incompatible with thoroughness, care, and rigor.

It is natural to raise the question of the potential readership of this book. Mathematical readers seeking insight into contemporary physics will probably be as much frustrated as enlightened. On the other hand, ultra-sophisticated mathematical physicists may be interested in the recasting of physical ideas in a very general mathematical framework, but they will probably be disappointed that no immediate solutions to their problems are forthcoming. Of course, most contemporary mathematical work is vulnerable to the same criticism. Certainly a very important aspect of the mathematician's job is the clarification, abstraction, and unification of ideas from a welter of confusing special cases; but unless the new insight yields practical results it will not be taken seriously (except by other mathematicians). Probably many of Hermann's ideas do have such "practical" value, but they will have to undergo further development before such value is demonstrated.

We point out that the faults of this series of books are much more prominent in LA than in some of the others; for example, MP and PS seem better organized and more clearly written. Also, VB-1 and VB-2, to which we now turn, seem to have better presentations on the whole.

Vector bundles in mathematical physics. I. In VB-1 Hermann treats the following topics: In Chapters I–V he discusses the calculus of variations and Lagrangian systems (in terms of jet bundles). Also, he deals with related ideas centering around Yang-Mills fields, conserved currents, and canonical quantum field theory. Chapter VI treats vector bundles and representations of semidirect products in the spirit of Mackey's ideas about induced representations. This is applied in Chapters VII and VIII to representations of the Lorentz group and Poincaré invariant operators. Finally, in Chapter IX, Fock space and free quantum fields are considered. Thus the content falls into three basically independent parts.

The material in Chapters I and II is fairly standard and is presented quite clearly. (However, it is treated in more detail and depth in PS.) From an abstract point of view, some calculations are done incorrectly. Specifically, Hermann makes the point that the classical expression $\int L(x, \phi(x), \partial\phi(x))\, dx$ is *defined* by integrating a function L on the jet bundle. This is fine, but he then calculates with this classical expression to obtain the Euler-Lagrange differential operator

$$\phi(x) \mapsto L_a(x, \phi(x), \partial\phi(x)) - \partial_\mu L_{a\mu}(x, \phi(x), \partial\phi(x))$$

(the author admits that this is not "covariant"). On a general vector bundle this expression and the calculations leading up to it are not well defined; one cannot integrate by parts in charts when the objects have no intrinsic meaning, i.e., are not tensorial. However, the result can be patched up by using a connection on the bundle and replacing derivatives by covariant derivatives where appropriate. This flaw is not serious and may indeed be avoidable, but it is illustrative of the sort of difficulty researchers and students encounter when trying seriously to understand the material.

Chapter III on Yang-Mills fields (see also FA, Chapter VII) contains a number of interesting ideas. In particular Hermann discusses the mathematical description of such fields in terms of connections on vector bundles. The title, however, lures the mathematician into thinking he will understand some physics at the end. But he will find only the following short discussion:

> ... One recognizes that (7.12) is obtained from (7.11) by what physicists call a "minimal electromagnetic interaction", i.e. $\partial_\mu \to (\partial_\mu - A_\mu)$, where $A_\mu(x)$ are the "vector potentials" of the electromagnetic field. Thus, we may say that, in the general case, the functions $(\alpha_{ab}(x), \alpha_{a\mu b}(x))$ are the "potentials" of a more complicated field, which has some sort of similarity—by generalization to the electromagnetic field. These more general "fields" are usually called—as some sort of generic name—"Yang-Mills fields".

As with LA the mathematician is faced with the task of trying to figure this out from the physics literature. The fact that here Hermann cites only other books of his does not make the task simpler. On the other hand, a physicist patient enough to master all the terminology required to understand this chapter would probably be disappointed at the end product.

Chapter V consists of a quite readable account of infinite dimensional symplectic forms and canonical quantum field theory. Some references to the literature (e.g. to Segal's book *Mathematical problems of relativistic physics*, Amer. Math. Soc., Providence, R.I., 1963) probably would have been helpful. The material actually seems to be better understood in the literature than this chapter might lead one to believe. Nevertheless, it is a good introductory account.

In Chapters VI–VIII we move to group-theoretical material. Chapter VI discusses Mackey's treatment of induced representations. The last section of the chapter sets up the Poincaré group, hints at its representations, but does not actually work them out—although they are mentioned and discussed later. Chapters VII and VIII study Poincaré invariant differential operators. These results seem to be slightly disorganized and not too clear. Indicative of the random way in which the author introduces terms, and especially odd following all the heavy machinery that he has used, is the author's taking time out on p. 295 to define the dual of a vector space and the adjoint of a linear transformation. Also, on pp. 325–326, he defines, at this late stage, "irreducible", yet assumes the reader knows the meaning of "semisimple". (The latter term is defined in Volume II!)

The final chapter contains a readable and elementary account of Fock space and free quantum fields. The author, as usual, emphasizes geometry at the expense of analysis. For example, the exact definition of Fock space as a *complete* Hilbert space and the nontrivial and important problems with unbounded operators are ignored. Again the mathematical reader is frustrated by the vague discussion of the physics. More references to the physics literature might have helped.

Vector bundles in mathematical physics. II. Volume 2 of VB is subtitled "Applications to Quantum Mechanics". This seems to mean applying vector bundle ideas, especially the idea of a differential operator in vector bundle language, to situations related to quantum mechanics. Chapter I deals with notational preliminaries. Chapter II gives an account of symplectic structures and classical mechanics. Possibly because of his aversion to hard analysis, the author leaves one with the impression that in the infinite dimensional case (see §§5, 6) there are serious analytic difficulties to be resolved. In fact, the situation seems to be more or less understood, although admittedly its treatment requires some technical machinery and is not dealt with in complete detail anywhere (cf., however, the Segal reference above).

Chapters III and IV are titled "Some basic theorems on Lie algebra theory" and "Homology and cohomology of manifolds". The author readily admits that these are quite out of place in this volume. We have here an illustration of the overall organizational difficulty from which this series suffers. We shall not comment on these chapters.

Chapter V is concerned with quantization and is closely connected with the author's ideas in LA discussed above. The choice of topics and emphasis is rather hard to justify. Although the author discusses the so-called van Hove representation, he does not emphasize one of van Hove's main results, which asserts that there is no physically reasonable general method to quantize every classical system (this fact is neatly hidden away on p. 306 in Chapter VII). Other results on intrinsically representing the maps $q^j \mapsto iq^i$ and $p^j \mapsto i(\partial/\partial q^j)$ on manifolds (which go back to Segal in 1960—but this is not mentioned) are also given; for some reason (unknown to the reviewers) the author requires the manifolds to be real analytic here. See also Chapter VI, §5. The vast amount of recent work on quantization due to Souriau and Kostant—an outgrowth of van Hove's work—is not mentioned, although it would have fit in nicely (cf. J. M. Souriau, *Structure des systemes dynamiques*, Dunod, 1970 for a summary of Souriau's work). In fact, many of these ideas, in a less developed form, seem to have been rediscovered by the author (see

Chapter VII, especially §§3, 4), although this may be just our interpretation as a result of inadequate referencing. §6 of Chapter V gives a short exposition of the passage from quantum to classical mechanics (see also Chapter VII), presenting an interesting geometrical way to make the well-known connections between the Hamilton-Jacobi and Schrodinger equations. In §7 the hydrogen atom and the conformal group are discussed. More complete and up to date accounts are available. (See for instance the articles in *Group theory and its applications*, E. M. Loebl (ed.), Academic Press, N.Y., 1971.) In §8 the Majorana representation is discussed. Again various mysterious statements are made which frustrate a mathematician, such as a reference to the "Gell-Mann formula"—frustrating, unless one is willing to do a lot of outside reading.

Chapter VI is titled "The symbol of a linear differential operator and its physical applications". The standard definitions and properties of the symbol are presented in the first three sections. §4 deals with Poincaré-invariant wave equations, and seems out of place here; the connection with symbols is minimal, and the ideas presented really belong in Chapter VII of VB-1. In §5, Hermann returns to the study of canonical quantization, and demonstrates that the symbol gives a homeomorphism-like map from differential operators on a manifold to functions on its cotangent bundle which are polynomials in the fiber variables.

Chapter VII, "Further work on the transition between classical and quantum mechanics", begins with a method for obtaining the classical limits of quantum systems more general than those discussed in Chapter V. §§4 and 5 are related to the recent work of Souriau, Kostant, Maslov, Arnold, and Hörmander (see also the comments on Chapter V). The presentation contains some of the basic ideas but, on the whole, is disorganized. This material is the subject of much current work, and the interested reader should consult the research papers.

Chapter VIII gives an all too brief discussion of mechanics on homogeneous spaces. The rotating top, treated here, is also discussed in the last chapter of DG.

Finally, Chapter IX takes up the theme of Galilean invariance of differential operators. It is a nice introductory account; for further results the serious student may wish to also consult J. M. Levy-Leblond's article in Volume II of *Group theory and its applications* (E. Loebl, ed.).

In conclusion, although they suffer to some extent from the defects of style and organization that plague LA, VB-1, 2 seem to be good books with many worthwhile ideas. Followers of mathematical physics will find Hermann's works stimulating, exhausting, and sometimes exasperating. But much of the literature of contemporary physics is even more exasperating for a mathematical reader; Hermann's books, despite their flaws, provide valuable clues to aid mathematicians to decipher the physicists' code. Hermann's contributions should further the growth and understanding of the vast, difficult, and exciting field situated in the frontier region between two flourishing sciences.

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