BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 79, Number 5, September 1973

A SERIES CHARACTERIZATION OF SUBSPACES OF $L_p(\mu)$ SPACES

BY NIGEL J. KALTON AND WILLIAM H. RUCKLE

Communicated by Robert Bartle, March 29, 1973

ABSTRACT. A normed space E is not isomorphic to a subspace of some $L_p(\mu)$ space if and only if there exists a series in E which does not converge absolutely but such that every continuous linear image of this series in l_p converges absolutely.

In this paper we derive the following theorem which is a strengthening of the Dvoretzky-Rogers theorem [1].

THEOREM. *A normed space E is not isomorphic to a subspace of a space $L_p(\mu)$ for any measure μ $(1 \leqq p \leqq \infty)$ if and only if* (*) *there exists in E a series $\sum_n x_n$ with each x_n in E such that $\sum_n \|x_n\| = \infty$ but $\sum_n \|Tx_n\| < \infty$ for each T in $L(E, l_p)$.*

The theorem is vacuously true for $p = \infty$ since every normed space is isometric to a subspace of the space of all bounded functions on some set.

The method used to prove this theorem is the analysis of the duality of vector sequence spaces. An account of this method is given in [3]. The proof encompasses normed spaces over both the real and complex fields.

For E a normed space we consider three spaces of sequences.

$l^1(E)$ consists of all (x_n) in E for which

$$\|(x_n)\| = \sum_n \|x_n\| < \infty.$$

$m(E)$ consists of all (x_n) in E for which

$$\|(x_n)\|_m = \sup\{\|x_n\| : n = 1, 2, \ldots\} < \infty.$$

$\sigma_p(E)$ consists of all sequences (x_n) in E for which

$$\|(x_n)\|_p = \sup\left\{\sum_n \|Tx_n\| : T \in U(E, l_p)\right\} < \infty.$$

Here $U(E, l_p)$ denotes the closed unit ball of $L(E, l_p)$ i.e. all continuous linear mappings T from E into l_p with $\|T\| \leqq 1$.

In the sequel we let U_p denote the closed unit ball of l_p and U_p° the polar

AMS (MOS) subject classifications (1970). Primary 46B15; Secondary 46C10.
Key words and phrases. Absolutely summable, Dvoretzky-Rogers theorem.

of U_p i.e. all continuous linear functionals x' on l_p such that $|\langle x, x'\rangle| \leqq 1$ for each $x \in U_p$.

It is well known that $l^1(E)$ and $m(E)$ are normed spaces with their respective norms $\| \ \|$ and $\| \ \|_m$. It is easy to see that $\sigma_p(E)$ is also a normed space with the norm $\| \ \|_p$. If E is a Banach space then all three of the vector sequence spaces are complete as well.

If E' is the topological dual space of E then $m(E')$ is isometric to the topological dual space of $l^1(E)$ under the natural bilinear form

$$\langle (x_n), (x_n')\rangle = \sum_n \langle x_n, x_n'\rangle \qquad (x_n) \in l^1(E), (x_n') \in m(E').$$

PROOF OF THE THEOREM. $1 \leqq p < \infty$.

NECESSITY OF (*). We may assume E is a complete space since if we can find a series in $\tilde{E}$, the completion of E, which satisfies (*) we can also find such a series in E by an easy perturbation argument.

If (*) does not hold then $l(E) = \sigma_p(E)$. Since both spaces $l(E)$ and $\sigma_p(E)$ are Banach spaces with their respective norms there is $\lambda > 0$ such that

$$\|(x_n)\| \leqq \lambda \|(x_n)\|_p, \tag{1}$$

from which we get

$$\|(x_n)\| \leqq \lambda \sup\left\{\sum_n \phi_n(Tx_n) : \phi_n \in U_p^\circ, T \in U(E, l_p)\right\}. \tag{2}$$

From (2) it follows that the unit ball of $m(E')$ is contained in the w^*-closed convex cover of sequences having the form $(\lambda T'\phi_n)$ where $\|T\| \leqq 1$ and $\|\phi_n\| \leqq 1$ for each n.

Suppose $A = \{x_1, x_2, \ldots, x_k\}$ is any finite subset of E. We shall show how to find a mapping T_A in $U(E, l_p)$ such that

$$\|T_A x\| > (1/2\lambda)\|x\| \qquad x \in A. \tag{3}$$

Let $X = (x_1', x_2', \ldots, x_k', 0, 0, \ldots)$ be a sequence in $m(E')$ such that $x_n'(x_n) = \|x_n\|$ and $\|x_n'\| = 1$ for $n = 1, 2, \ldots, k$. By the preceding paragraph we can find $T_1, \ldots, T_r$ in $U(E, l_p)$; c_i, $i = 1, 2, \ldots, r$ with $\sum_{i=1}^r c_i = 1$ and ϕ_{ij}, $i = 1, 2, \ldots, r; j = 1, 2, \ldots$ in U_p° such that

$$\left|\left\langle X - \sum_{i=1}^r c_i(\lambda T_i'\phi_{ij}), x_n e_n\right\rangle\right| < \tfrac{1}{2}\min\{\|x_n\| : n = 1, 2, \ldots, k\} \tag{4}$$

for each x_n; $n = 1, 2, \ldots, k$. Here $x_n e_n$ is the sequence with x_n in the nth place and 0's elsewhere. From (4) we see that for each n

$$\left|\|x_n\| - \sum_{i=1}^r c_i\lambda\langle (T_r'\phi_{ij}), e_n x_n\rangle\right| < \tfrac{1}{2}\|x_n\|$$

or what is equivalent

$$\sum_{i=1}^{r} c_i \, \|T_i'\phi_{in}\| > \frac{1}{2\lambda} \|x_n\|; n = 1, 2, \ldots, k. \tag{5}$$

Let Z consist of all r-tuples $(y_1, \ldots, y_r)$ with each y_i in l_p. With the norm $|(y_i)| = (\sum_{i=1}^r c_i \|y_i\|^p)^{1/p}$, Z is isometric to l_p. Define T_A in $L(E, Z)$ by $T(x) = (T_i X)$. Then $\|T_A\| \leqq 1$ and since $\sum_{i=1}^r c_i = 1$

$$\|T_A x_n\| \geqq \sum_{i=1}^{r} |c_i| \, \|T_i x_n\| > \frac{1}{2\lambda} \|x_n\|; \qquad n = 1, 2, \ldots, k.$$

Thus T_A satisfies (3).

If F is any finite dimensional subspace of E and $0 < \varepsilon < 1/2\lambda$ arbitrary let A be an ε-net for the unit sphere of F. If T_A in $U(E, l_p)$ satisfies (3) then T_F the restriction of T_A to F is an isomorphism from F into l_p with $\|T_F\| \, \|T_F^{-1}\| \leqq 1/2\lambda - \varepsilon$. Thus by Proposition 7.1 of [**2**] E is isomorphic to a subspace of $L_p(\mu)$ for some measure μ.

SUFFICIENCY OF (*). It suffices to show that there is $\lambda > 0$ such that if A is any finite subset of $L_p(\mu)$ there is $T_A \in L(L_p(\mu), l_p)$ with

$$\sum_{x \in A} \|T_A x\| \leqq \lambda \sum_{x \in A} \|T_A(x)\|.$$

The dual space $(L_p(\mu))'$ of $L_p(\mu)$ is an $\mathscr{L}'_p$ space so there is a continuous linear isomorphism S from $l^k_{p'}$ into $(L_p \mu)$ whose image contains $\{y'_x : x \in A\}$ where $\langle x, y'_x \rangle = \|x\|$ for each $x \in A$, and $\|S\| \, \|S^{-1}\| < \lambda$ where λ is independent of A. We can assume $\|S\| \leqq 1$ and extend S to all of $l_{p'}$ by means of the natural projection of $l_{p'}$ onto $l^k_{p'}$. Let ϕ_x in $l_{p'}$ be such that $\|\phi_x\| \leqq \lambda$ and $S\phi_x = y'_x$. Then

$$\begin{aligned} \sum_{x \in A} \|x\| &= \sum_{x \in A} \langle x, y'_x \rangle = \sum_{x \in A} \langle x, S\phi_x \rangle \\ &\leqq x \sum_{x \in A} \|S'x\|. \end{aligned}$$

We complete the proof by taking T_A to be S' (restricted to $L_p(\mu)$ in the case $p = 1$).

If we set p equal to 2 in the theorem we obtain the following characterization of Hilbert space.

COROLLARY. *A normed space E is not isomorphic to an inner product space if and only if there exists in E a series $\sum_n x_n$ with $\sum_n \|x_n\| = \infty$ but $\sum_n \|Tx_n\| < \infty$ for each T from E into Hilbert space.*

The Dvoretzky-Rogers theorem follows easily from this corollary. In fact, if E is an infinite dimensional normed space which is isomorphic to

Hilbert space we let $x_n = (1/n)y_n$ where $\{y_n\}$ is an orthonormal system in E with respect to some inner product. Otherwise we let $\{x_n\}$ be obtained by the corollary. In either case $\sum_n x_n$ converges unconditionally but not absolutely.

REFERENCES

1. A. Dvoretzky and C. A. Rogers, *Absolute and unconditional convergence in normed linear spaces*, Proc. Nat. Acad. Sci. U.S.A. **36** (1950), 192–197. MR **11**, 525.
2. J. Lindenstrauss and A. Pełczyński, *Absolutely summing operators in L_p spaces and their applications*, Studia Math. **29** (1968), 275–326. MR **37** #6743.
3. W. Ruckle, *Decompositions of operator spaces*, Rev. Roumaine Math. Pures Appl. **15** (1970), 119–134. MR **41** #8975.

DEPARTMENT OF MATHEMATICAL SCIENCES, CLEMSON UNIVERSITY, CLEMSON, SOUTH CAROLINA 29631 (Current address of William H. Ruckle)

DEPARTMENT OF MATHEMATICS, UNIVERSITY COLLEGE AT SWANSEA, SWANSEA, WALES (Current address of Nigel J. Kalton)