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THE COMPREHENSIVE FACTORIZATION OF A FUNCTOR

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In this article we show that every functor has a factorization into an initial functor followed by a discrete 0-fibration and that this factorization is functorial. Size considerations will be ignored but may be easily filled in; we assume the existence of a category of sets large enough to dwarf any given finite number of categories.

There is an analogy between the category **Set** of sets and the category **Cat** of categories which is partly explained by the observation that each is a category of types for a suitable hyperdoctrine. A hyperdoctrine (Lawvere [**4**]) consists of a category T of types and a functor $P: T^{\mathrm{op}} \to \mathbf{Cat}$ satisfying conditions. The comprehension schema (also see [**4**]) is expressed by a pair of adjoint functors

$$PX \leftrightarrows T/X$$

for each object X of T, where T/X is the category of objects over X. It is often the case that this structure arises from more usual structure on the category T; namely:

(1) a factorization system (E, M) on T;

(2) a category object Ω in T which "classifies the M-subobjects".

The sense in which (1) is intended is that of Freyd-Kelly [**2**]. We say that Ω classifies the M-subobjects when there is a natural equivalence of categories $T(X, \Omega) \approx M(X)$, where $M(X)$ is the full subcategory of T/X consisting of the arrows in M with target X. Then $PX = T(X, \Omega)$. From (1), the functor "take the (E, M)-image" is the left adjoint of the inclusion $M(X) \to T/X$; this adjunction combines with (2) to yield the comprehension schema.

The familiar example of a hyperdoctrine which arises in the above way is provided by the power-set functor $P: \mathbf{Set}^{\mathrm{op}} \to \mathbf{Cat}$. Here M consists of monomorphisms, E of epimorphisms, and Ω is a set with two elements. These considerations lie at the heart of the elementary theory of the category of sets in the new elegant form—elementary topos theory—due to Lawvere-Tierney (see Freyd [**1**]). A topos T is a finitely complete, cartesian closed category satisfying (2) where M consists of the mono-

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morphisms. It can be proved that (E, M) is a factorization system on T where E consists of the epimorphisms.

It is our aim to show here that there is a factorization system on **Cat** which gives rise to the hyperdoctrine with $T = \mathbf{Cat}$ and PX equal to the category of set-valued functors on X. Moreover, we show that the classes E, M are already distinguished in the literature; namely, E consists of the initial functors, M consists of the discrete 0-fibrations. Forthcoming papers (see [6]) on the elementary theory of the 2-category of categories will consider the abstraction of the concept of discrete 0-fibration to an arbitrary 2-category and the corresponding condition (2).

For any category A, let $!: A \to \mathbf{1}$ denote the unique functor into the terminal object of **Cat**. Let $*: \mathbf{1} \to \mathbf{Set}$ denote the constant functor at the one point set. For functors $A \to^r D$, $B \to^s D$ with the same codomain, the *comma category* r/s is defined by the pullback

$$\begin{tikzcd} r/s \arrow[r] \arrow[d, "\binom{d_0}{d_1}"'] & D^2 \arrow[d, "\binom{d_0}{d_1}"] \\ A \times B \arrow[r, "r \times s"'] & D \times D. \end{tikzcd}$$

The natural transformation

$$\begin{tikzcd} r/s \arrow[r, "d_1"] \arrow[d, "d_0"'] & B \arrow[d, "s"] \\ A \arrow[r, "r"'] \arrow[ur, phantom, "\lambda"] & D \end{tikzcd}$$

corresponding to $r/s \to D^2$ is characterized by the universal property that composition with λ sets up a bijection

$$\begin{tikzcd} & X \arrow[dl, "u"'] \arrow[dr, "v"] \arrow[d, "w"] & & & X \arrow[r, "v"] \arrow[d, "u"'] & B \arrow[d, "s"] \\ A & r/s \arrow[l, "d_0"] \arrow[r, "d_1"'] & B & \longleftrightarrow & A \arrow[r, "r"'] \arrow[ur, phantom, "\sigma"] & D \end{tikzcd}$$

between arrows of spans w and natural transformations σ. The particular case which is important here is the comma category $*/k$ for a functor $B \to^k \mathbf{Set}$; the objects are pairs (b, ξ) where b is an object of B and ξ is an element of kb, and the arrows $(b, \xi) \to^\beta (b', \xi')$ are arrows $b \to^\beta b'$ in B satisfying $(k\beta)\xi = \xi'$.

A natural transformation

$$\begin{tikzcd} A \arrow[rr, "j"] \arrow[dr, "f"'] & \arrow[d, phantom, "\curvearrowright"] & B \arrow[dl, "k"] \\ & \mathbf{Set} & \end{tikzcd}$$

is said to exhibit k as a *left extension* of f along j when it sets up a bijection between natural transformations

$$\begin{tikzcd} B \arrow[r, bend left, "k"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\downarrow"] & \mathbf{Set} \end{tikzcd}$$

and natural transformations

$$\begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, bend right, "gj"'] \arrow[r, phantom, "\downarrow"] & \mathbf{Set}. \end{tikzcd}$$

Recall also (see Mac Lane [5, p. 236]) the formula

$$kb = \operatorname{colim}(j/b \xrightarrow{d_0} A \xrightarrow{f} \mathbf{Set}).$$

PROPOSITION 1. *For any functor $B \to^k$ **Set**, the natural transformation*

$$\begin{tikzcd} */k \arrow[r, "d_1"] \arrow[d, "! = d_0"'] & B \arrow[d, "k"] \\ \mathbf{1} \arrow[r, "*"'] \arrow[ur, phantom, "\lambda"] & \mathbf{Set} \end{tikzcd}$$

exhibits k as a left extension of $*!$ along d_1. □

A functor $E \to^p B$ is called a *discrete* 0-*fibration* when there exists a functor $B \to^k$ **Set** and an isomorphism $E \cong */k$ such that

$$\begin{tikzcd} E \arrow[rr, phantom, "\cong"] \arrow[dr, "p"'] & & */k \arrow[dl, "d_1"] \\ & B & \end{tikzcd}$$

commutes. By Proposition 1, k is uniquely determined up to isomorphism by p.

A functor $A \to^e E$ is called *initial* when the canonical natural transformation

$$\begin{tikzcd} \mathbf{Set}^E \arrow[rr, "\mathbf{Set}^e"] \arrow[dr, "\lim"'] & & \mathbf{Set}^A \arrow[dl, "\lim"] \arrow[dl, phantom, bend left, "\mathrm{res}_e"'] \\ & \mathbf{Set} & \end{tikzcd}$$

is an isomorphism.

These two definitions can be made independent of the choice of the category **Set**. A functor $E \to^p B$ is called a *0-fibration* (Gray [3]) when the canonical functor $E^2 \to p/B$, induced by the natural transformation

$$\begin{tikzcd} E^2 \arrow[r, "pd_1"] \arrow[d, "d_0"'] & B \arrow[d, "1"] \\ E \arrow[r, "p"'] \arrow[ur, phantom, "p\lambda"] & B, \end{tikzcd}$$

has a left adjoint with identity unit. The 0-fibration p is discrete if and only if, for each object b in B, the category E_b, defined by the pullback

$$\begin{tikzcd} E_b \arrow[r] \arrow[d] & E \arrow[d, "p"] \\ \mathbf{1} \arrow[r, "b"'] & B \end{tikzcd}$$

is discrete. This is easily proved by defining $B \to^k \mathbf{Set}$ on objects by $kb = E_b$. That (a), (c) in the following proposition are equivalent is well known (see Mac Lane [**5** p. 213] for the dual); we give a different proof. Condition (b) is the important one for us.

PROPOSITION 2. *The following conditions are equivalent*:
(a) *the functor $A \to^e E$ is initial*;
(b) *the identity natural transformation*,

$$\begin{tikzcd} A \arrow[rr, "e"] \arrow[dr, "*!"'] & & E \arrow[dl, "*!"] \\ & \mathbf{Set} \arrow[u, phantom, "\xrightarrow{1}"] & \end{tikzcd}$$

exhibits $*!$ as a left extension of $*!$ along e;
(c) *for each object x in E, the comma category e/x is nonempty and pathwise connected.*

PROOF. Consider the natural transformation in the triangle

$$\begin{tikzcd} \mathbf{Set}^E & & \mathbf{Set}^A \arrow[ll, "\Sigma_e"'] \\ & \mathbf{Set} \arrow[ul, "\Delta"] \arrow[ur, "\Delta"'] \arrow[u, phantom, "\curvearrowleft"] & \end{tikzcd}$$

obtained from the triangle containing res_e by replacing each functor by its left adjoint. Then Δ takes a set to the constant functor at that set, and Σ_e takes a functor to its left extension along e. Also e is initial if and only if $\Sigma_e\Delta \to \Delta$ is an isomorphism. Every constant functor $A \to \mathbf{Set}$ is a coproduct of copies of the constant functor $*!$ and Σ_e, being a left adjoint, preserves coproducts. So $\Sigma_e\Delta \to \Delta$ is an isomorphism if and only if $\Sigma_e(*!) = *!$. This proves (a) $\Leftrightarrow$ (b). From the construction of colimits in **Set** it is clear that (c) says precisely that, for each object x in E,

$$\operatorname{colim}(e/x \xrightarrow{!} \mathbf{1} \xrightarrow{*} \mathbf{Set})$$

is the one point set. Using the formula for left extensions, we see that this is equivalent to (b). $\square$

THEOREM 3. *Any functor $A \to^f B$ admits a factorization*

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "e"'] & & B \\ & E \arrow[ur, "p"'] & \end{tikzcd}$$

where e is an initial functor and p is a discrete 0*-fibration.*

PROOF. Define $B \to^k$ **Set** to be the left extension of $A \to^! \mathbf{1} \to^* \mathbf{Set}$ along f. Then define $E \to^p B$ to be $*/k \to^{d_1} B$. So p is a discrete 0-fibration. The natural transformation

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "!"'] & B \arrow[d, "k"] \\ \mathbf{1} \arrow[r, "*"'] \arrow[ur, phantom, "\Rightarrow"] & \mathbf{Set} \end{tikzcd}$$

factors through the natural transformation

$$\begin{tikzcd} E \arrow[r, "p"] \arrow[d, "!"'] & B \arrow[d, "k"] \\ \mathbf{1} \arrow[r, "*"'] \arrow[ur, phantom, "\Rightarrow"] & \mathbf{Set} \end{tikzcd}$$

by the universal property of the latter. So we have a functor e such that

$$\begin{tikzcd} & A \arrow[dl] \arrow[dr, "f"] \arrow[d, "e"'] & \\ \mathbf{1} & E \arrow[l] \arrow[r, "p"'] & B \end{tikzcd}$$

commutes. Consider the diagram

$$\begin{tikzcd} A \arrow[r, "e"] \arrow[dr, "*!"'] & E \arrow[r, "p"] \arrow[d, "*!"] \arrow[dl, phantom, "\Leftarrow"] \arrow[dr, phantom, "\Rightarrow"] & B \arrow[dl, "k"] \\ & \mathbf{Set} & \end{tikzcd}$$

The outside triangle is a left extension by definition of k. The right triangle is a left extension by Proposition 1. So the left triangle is a left extension; e is initial. $\square$

THEOREM 4. *Suppose, in the commutative diagram*

$$\begin{tikzcd} A \arrow[r, "e"] \arrow[d, "u"'] & C \arrow[d, "v"] \\ E \arrow[r, "p"'] & B \end{tikzcd}$$

of functors, e is initial and p is a discrete 0-fibration. Then there exists a unique functor $C \to^{w} E$ such that $pw = v$ and $we = u$.

PROOF. Since p is a discrete 0-fibration there is a functor k and a universal natural transformation

$$\begin{tikzcd} E \arrow[r, "p"] \arrow[d, "!"'] & B \arrow[d, "k"] \\ \mathbf{1} \arrow[r, "*"'] \arrow[ur, phantom, "\lambda"] & \mathbf{Set}. \end{tikzcd}$$

Since e satisfies (b) of Proposition 2, composing with e sets up a bijection

$$\begin{tikzcd} C \arrow[r, "v"] \arrow[d, "!"'] & B \arrow[d, "k"] & \longleftrightarrow & A \arrow[r, "ve"] \arrow[d, "!"'] & B \arrow[d, "k"] \\ \mathbf{1} \arrow[r, "*"'] \arrow[ur, phantom, "\alpha"] & \mathbf{Set} & & \mathbf{1} \arrow[r, "*"'] \arrow[ur, phantom, "\beta"] & \mathbf{Set} \end{tikzcd}$$

between natural transformations α and natural transformations β. By the universal property of λ, this implies that composing with e sets up a bijection

$$\begin{tikzcd} & C \arrow[dl] \arrow[dr, "v"] \arrow[dd, "w"'] & & & & A \arrow[dl] \arrow[dr, "ve"] \arrow[dd, "u"'] & \\ \mathbf{1} & & B & \longleftrightarrow & \mathbf{1} & & B \\ & E \arrow[ul] \arrow[ur, "p"'] & & & & E \arrow[ul] \arrow[ur, "p"'] & \end{tikzcd}$$

between arrows of spans w and arrows of spans u. □

It follows from Theorem 4 that the factorization of Theorem 3 is unique up to isomorphism and that the notion of image so obtained is functorial.

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