THE COMPREHENSIVE FACTORIZATION OF A FUNCTOR

BY ROSS STREET AND R. F. C. WALTERS

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In this article we show that every functor has a factorization into an initial functor followed by a discrete 0-fibration and that this factorization is functorial. Size considerations will be ignored but may be easily filled in; we assume the existence of a category of sets large enough to dwarf any given finite number of categories.

There is an analogy between the category **Set** of sets and the category **Cat** of categories which is partly explained by the observation that each is a category of types for a suitable hyperdoctrine. A hyperdoctrine (Lawvere [4]) consists of a category T of types and a functor $P: T^{op} \to \mathbf{Cat}$ satisfying conditions. The comprehension schema (also see [4]) is expressed by a pair of adjoint functors

$$PX \Leftrightarrow T/X$$

for each object X of T, where T/X is the category of objects over X. It is often the case that this structure arises from more usual structure on the category T; namely:

- (1) a factorization system (E, M) on T;
- (2) a category object Ω in T which "classifies the M-subobjects".

The sense in which (1) is intended is that of Freyd-Kelly [2]. We say that Ω classifies the M-subobjects when there is a natural equivalence of categories $T(X,\Omega) \approx M(X)$, where M(X) is the full subcategory of T/X consisting of the arrows in M with target X. Then $PX = T(X,\Omega)$. From (1), the functor "take the (E,M)-image" is the left adjoint of the inclusion $M(X) \to T/X$; this adjunction combines with (2) to yield the comprehension schema.

The familiar example of a hyperdoctrine which arises in the above way is provided by the power-set functor $P:\mathbf{Set^{op}}\to\mathbf{Cat}$. Here M consists of monomorphisms, E of epimorphisms, and Ω is a set with two elements. These considerations lie at the heart of the elementary theory of the category of sets in the new elegant form—elementary topos theory—due to Lawvere-Tierney (see Freyd [1]). A topos T is a finitely complete, cartesian closed category satisfying (2) where M consists of the mono-

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morphisms. It can be proved that (E, M) is a factorization system on T where E consists of the epimorphisms.

It is our aim to show here that there is a factorization system on Cat which gives rise to the hyperdoctrine with T = Cat and PX equal to the category of set-valued functors on X. Moreover, we show that the classes E, M are already distinguished in the literature; namely, E consists of the initial functors, M consists of the discrete 0-fibrations. Forthcoming papers (see [6]) on the elementary theory of the 2-category of categories will consider the abstraction of the concept of discrete 0-fibration to an arbitrary 2-category and the corresponding condition (2).

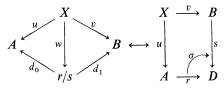
For any category A, let $!:A \to 1$ denote the unique functor into the terminal object of Cat. Let $*:1 \to Set$ denote the constant functor at the one point set. For functors $A \to {}^r D$, $B \to {}^s D$ with the same codomain, the comma category r/s is defined by the pullback

$$\begin{array}{ccc}
r/s & \longrightarrow & D^2 \\
\begin{pmatrix} d_0 \\ d_1 \end{pmatrix} \downarrow & & \downarrow \begin{pmatrix} d_0 \\ d_1 \end{pmatrix} \\
A \times B \xrightarrow{r \times s} D \times D.$$

The natural transformation

$$\begin{array}{ccc}
r/s & \xrightarrow{d_1} & B \\
\downarrow d_0 & & \downarrow s \\
A & \xrightarrow{r} & D
\end{array}$$

corresponding to $r/s \to D^2$ is characterized by the universal property that composition with λ sets up a bijection



between arrows of spans w and natural transformations σ . The particular case which is important here is the comma category */k for a functor $B \to^k \mathbf{Set}$; the objects are pairs (b, ξ) where b is an object of B and ξ is an element of kb, and the arrows $(b, \xi) \to^{\beta} (b', \xi')$ are arrows $b \to^{\beta} b'$ in B satisfying $(k\beta)\xi = \xi'$.

A natural transformation

$$\begin{array}{c}
A \xrightarrow{j} B \\
f \searrow k
\end{array}$$

is said to exhibit k as a *left extension* of f along j when it sets up a bijection

between natural transformations

$$B \underbrace{\downarrow}_{a}^{k} \mathbf{Set}$$

and natural transformations

$$A \underbrace{\bigcup_{q_i}^f}_{\mathbf{Set}} \mathbf{Set}.$$

Recall also (see Mac Lane [5, p. 236]) the formula

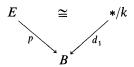
$$kb = \operatorname{colim}(j/b \xrightarrow{d_0} A \xrightarrow{f} \mathbf{Set}).$$

PROPOSITION 1. For any functor $B \to^k \mathbf{Set}$, the natural transformation

$$\begin{array}{ccc}
*/k & \xrightarrow{d_1} & B \\
! & = d_0 & & \downarrow k \\
1 & \xrightarrow{*} & \mathbf{Set}
\end{array}$$

exhibits k as a left extension of *! along d_1 . \square

A functor $E \to^p B$ is called a *discrete* 0-fibration when there exists a functor $B \to^k \mathbf{Set}$ and an isomorphism $E \cong */k$ such that



commutes. By Proposition 1, k is uniquely determined up to isomorphism by p.

A functor $A \rightarrow^{e} E$ is called *initial* when the canonical natural transformation

$$\underbrace{\operatorname{Set}^{E} \quad \underbrace{\operatorname{Set}^{e}}_{\operatorname{res}_{e}} \quad \operatorname{Set}^{A}}_{\operatorname{Set}}$$

is an isomorphism.

These two definitions can be made independent of the choice of the category **Set**. A functor E o p B is called a 0-fibration (Gray [3]) when the canonical functor $E^2 o p/B$, induced by the natural transformation

$$E^{2} \xrightarrow{pd_{1}} B$$

$$\downarrow d_{0} \downarrow p\lambda \qquad \downarrow 1$$

$$E \xrightarrow{p} B,$$

has a left adjoint with identity unit. The 0-fibration p is discrete if and only if, for each object b in B, the category E_b , defined by the pullback

$$\begin{array}{ccc}
E_b & \longrightarrow & E \\
\downarrow & & \downarrow^p \\
\mathbf{1} & \xrightarrow{b} & B
\end{array}$$

is discrete. This is easily proved by defining $B \to^k \mathbf{Set}$ on objects by $kb = E_b$. That (a), (c) in the following proposition are equivalent is well known (see Mac Lane [5 p. 213] for the dual); we give a different proof. Condition (b) is the important one for us.

Proposition 2. The following conditions are equivalent:

- (a) the functor $A \rightarrow^e E$ is initial;
- (b) the identity natural transformation,

$$\begin{array}{c}
A \xrightarrow{e} E \\
*! \downarrow \downarrow / *! \\
\mathbf{Set}
\end{array}$$

exhibits *! as a left extension of *! along e;

(c) for each object x in E, the comma category e/x is nonempty and pathwise connected.

PROOF. Consider the natural transformation in the triangle

$$\mathbf{Set}^{E} \stackrel{\Sigma_{e}}{\longleftarrow} \mathbf{Set}^{A}$$

$$\Delta$$

$$\mathbf{Set}$$

obtained from the triangle containing res_e by replacing each functor by its left adjoint. Then Δ takes a set to the constant functor at that set, and Σ_e takes a functor to its left extension along e. Also e is initial if and only if $\Sigma_e \Delta \to \Delta$ is an isomorphism. Every constant functor $A \to \mathbf{Set}$ is a coproduct of copies of the constant functor *! and Σ_e , being a left adjoint, preserves coproducts. So $\Sigma_e \Delta \to \Delta$ is an isomorphism if and only if $\Sigma_e(*!) = *!$. This proves (a) \Leftrightarrow (b). From the construction of colimits in \mathbf{Set} it is clear that (c) says precisely that, for each object x in E,

$$\operatorname{colim}(e/x \stackrel{!}{\longrightarrow} \mathbf{1} \stackrel{*}{\longrightarrow} \mathbf{Set})$$

is the one point set. Using the formula for left extensions, we see that this is equivalent to (b). \Box

THEOREM 3. Any functor $A \rightarrow^f B$ admits a factorization



where e is an initial functor and p is a discrete 0-fibration.

PROOF. Define $B \to^k \mathbf{Set}$ to be the left extension of $A \to^! \mathbf{1} \to^* \mathbf{Set}$ along f. Then define $E \to^p B$ to be $*/k \to^{d_1} B$. So p is a discrete 0-fibration. The natural transformation



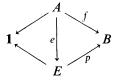
factors through the natural transformation

$$E \xrightarrow{p} B$$

$$\downarrow \downarrow \qquad \downarrow^{k}$$

$$1 \xrightarrow{*} \mathbf{Set}$$

by the universal property of the latter. So we have a functor e such that



commutes. Consider the diagram

$$A \xrightarrow{e} E \xrightarrow{p} B$$

$$*! \downarrow *! \downarrow k$$
Set

The outside triangle is a left extension by definition of k. The right triangle is a left extension by Proposition 1. So the left triangle is a left extension; e is initial. \square

THEOREM 4. Suppose, in the commutative diagram

$$\begin{array}{ccc}
A & \xrightarrow{e} & C \\
u \downarrow & & \downarrow v \\
E & \xrightarrow{p} & B
\end{array}$$

of functors, e is initial and p is a discrete 0-fibration. Then there exists a unique functor $C \to^w E$ such that pw = v and we = u.

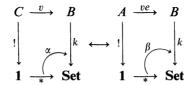
PROOF. Since p is a discrete 0-fibration there is a functor k and a universal natural transformation

$$E \xrightarrow{p} B$$

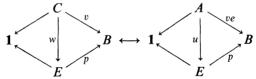
$$\downarrow \downarrow \qquad \downarrow^{k}$$

$$1 \xrightarrow{\longrightarrow} \mathbf{Set}.$$

Since e satisfies (b) of Proposition 2, composing with e sets up a bijection



between natural transformations α and natural transformations β . By the universal property of λ , this implies that composing with e sets up a bijection



between arrows of spans w and arrows of spans u. \square

It follows from Theorem 4 that the factorization of Theorem 3 is unique up to isomorphism and that the notion of image so obtained is functorial.

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DEPARTMENT OF MATHEMATICS, MACQUARIE UNIVERSITY, NORTH RYDE 2113, AUSTRALIA DEPARTMENT OF MATHEMATICS, UNIVERSITY OF SYDNEY, SYDNEY 2006, AUSTRALIA