ideal to be a certain collection of square matrices over a ring R, which is closed under two operations. It is possible to develop a theory of prime matrix ideals having essentially the properties of prime ideals of a ring. The main theorem states that given a prime matrix ideal $\mathscr{P}$ of a ring R there exists a field K and a homomorphism $R \to K$ such that $\mathscr{P}$ is precisely the class of matrices mapped to singular matrices under the homomorphism $R \to K$. The field K is obtained by "localizing" R at the "multiplicatively closed" set Σ consisting of all square matrices in the complement of $\mathscr{P}$.

This theorem is used to give a necessary and sufficient criterion for the embeddability of a ring into a skew field.

Chapter 8 is devoted to special properties of (noncommutative) P.I.D., for example, the diagonal reduction of matrices. In particular, every finitely generated module M over a P.I.D. R is a direct sum of cyclic modules:

$$M = R/e_1R \oplus \cdots \oplus R/e_rR \oplus {}^{m-r}R$$

where e_i is a total divisor of e_{i+1} for $i = 1, \ldots, r-1$ and this condition determines the e_i's up to similarity. (That e is a *total divisor* of e' means that there exists an element c of R such that $cR = Rc$ and e divides c and c divides e'.)

The results are applied to characterize the invariant elements of the skew polynomial ring $R = k[t; S, D]$ with automorphism S and S-derivation D, and also to investigate certain algebraic extensions of skew fields.

A. CZERNIAKIEWICZ

Stabilité Structurelle et Morphogénèse, Essai d'une Théorie Générale des Modèles, by René Thom. Benjamin, New York, 1971. 384 pp. $20.

René Thom has written a provocative book. It contains much of interest to mathematicians and has already had a significant impact upon mathematics, but *Stabilité Structurelle et Morphogénèse* is not a work of mathematics. Because Thom is a mathematician, it is tempting to apply mathematical standards to the work. This is certainly a mistake since Thom has made no pretense of having tried to meet these standards. He even ends the book with a plea for the freedom to write vaguely and intuitively without being ostracized by the mathematical community for doing so. Instead of insisting that Thom's style conform to prevailing norms, we should applaud him for sharing his wonderful imagination with us.

The book touches upon an enormous spectrum of material from developmental biology to optics to linguistics as well as mathematics. I can bring to the book only the perspective of a mathematician and consequently I am unable to assess the impact of Thom's theories in broad terms. In this review, therefore, I shall largely ignore the question of whether Thom presents an accurate description of those biological and linguistic phenomena with which he deals. Instead I shall focus upon the mathematics underlying his theories. While this aspect of his work is likely to excite more interest among mathematicians, it leaves aside other essential parts of the work. In this respect, I must apologize to Thom for being incapable of adequately discussing the whole book and to the reader for presenting an unbalanced commentary.

Thom states that his initial goal in constructing the theory was to give more precision to the concepts of "chreod" and "epigenetic landscape" created by the biologist C. H. Waddington. Waddington invented chreods to help explain some of the properties of developing organisms. Amphibia are perhaps the most common experimental material of embryologists, so let us briefly consider the development of the frog egg. At various stages in its development into a tadpole, the frog egg undergoes drastic changes of shape and form. One of the first of these changes is the process of gastrulation. When gastrulation begins, the embryo is roughly a spherical shell. On its surface a crescent grows from a point until an entire circle is formed. Along this crescent there is invagination of tissue into the interior of the spherical shell to form the primitive gut of the animal.

The problem of Thom is to describe the *geometry* of gastrulation and similar morphogenetic events in developing animals. Instead of attacking such a difficult problem directly, Thom constructs very general geometric models. These models are applicable to many other phenomena besides biological morphogenesis. Indeed, Thom asserts that they are fundamental in our perception of the world. The geometric models form a universal language shared by everyone. They are translated into a one dimensional linear structure by ordinary language. For Thom, geometric interpretation is almost the same thing as intellectualisation; consequently, it is through geometry that we attain an intuitive understanding of the world about us.

Underlying the entire theory is a fundamental hypothesis of structural stability. This stability manifests itself in a local constancy of qualitative structures. Thus two members of the same species can be quite far from being metrically congruent but qualitatively we sense their similarity. How can one express this kind of similarity? Thom suggests that ordinary language does it well. Indeed he asserts that structurally stability is inherent in language: it is impossible to describe structurally unstable processes in simple language. So Thom accepts structural stability as an a priori requirement for his models and examines the mathematical possibilities consistent with this hypothesis.

What is the structure of the models? Thom begins their construction in Chapter 1. As we look at the world, we associate to (almost) every point in space and time a state. There are regions of points which have states that can be labeled by names such as heart, water, or table. Inside these regions, the state of a point does not undergo a qualitative change. Those points at which the state locally changes continuously are called *regular* points. A *catastrophe* point is one which is not regular. One encounters here the problem of how one determines the state of a point. In order to apply geometric analysis to the models, it is assumed that the local state can be specified as a point in some Euclidean space or manifold. In terms of biological applications, the state of a cell is determined by measuring its physical and chemical properties.

Thus, the space in which we work is a manifold $M \to \boldsymbol{R}^4$ fibered over four dimensional space-time. $\boldsymbol{R}^4$ is the space of *external parameters* and the fiber of the fibration is the space of *internal parameters*. There is a (discontinuous) section of the fibration which associates to each point its state. The points of discontinuity of this section are called catastrophes. At this stage, the action of physical laws is not yet built into the system. This is done by assuming that there is a vector field X defined on M which is almost tangent to the fibers. As time proceeds, the state of a point follows the integral curves of X. If we examine a region in which the state does not change quickly, then the states of points in those regions will lie near an asymptotic limit of an orbit of X. In the simplest case, X is actually tangent to the fiber and the state of a point is near a stable singular point of X.

This is the general form of the model, but it still has little structure. We need to make the basic stability assumption in order to give some content to the theory. Chapter 2 introduces this assumption: the geometric structure of the catastrophe set should be stable with respect to perturbations of the vector field X and the section of M defining the state. Thom gives his justification for making this hypothesis of structural stability. Those concepts which it is possible to think about and communicate have an inherent stability insofar as the concept is the same each time we think of it. This epistemological stability should then be built into the models we construct of reality because it is so basic. Developing organisms also show strong stability properties and power of regulation. Thus, one can assert the structural stability of development as an experimental observation and construct a theory incorporating this assumption.

Chapters 3, 4 and 5 are primarily mathematical. Chapter 3 is a survey of the concept of genericity and structural stability in different areas of mathematics: algebraic and analytic geometry, differential topology, and ordinary and partial differential equations. The models used later in the book draw principally upon differential topology and ordinary differential equations. The general pattern is the following: let L be a function space with the structure of an infinite-dimensional manifold. The most relevant examples are the space of C^1 vector fields on a manifold and the space of smooth maps between two manifolds. Define an equivalence relation on L. With respect to this equivalence relation, an element of L which is an interior point of its equivalence class is said to be *stable*. The complement of the set of stable points in L is a closed set K called the *bifurcation set* of the equivalence relation. One hopes that K is nowhere dense. If this is the case, then every element of L can be approximated by one which is stable. One hopes further that K is a stratified set which can be decomposed $K = \bigcup_{i=1}^{\infty} K^i \cup H$ with K^i a submanifold of L of codimension i for $1 \leqq i < \infty$ and H a subset of $L$of infinite codimension. If K does admit such a decomposition, then a *universal unfolding* of $f \in K^i$ is a map $F:\boldsymbol{R}^i \to L$ such that $F(o) = f$ and F is transverse to each K^i. The stratification of L by the K^i then pulls back to a stratification of $\boldsymbol{R}^i$ via the map F. Under the additional hypotheses that (1) each component of K^i is contained in a single equivalence class, and (2) the stratification of K satisfies certain regularity properties, F has the universal property that given another map $G:\boldsymbol{R}^j \to L$ with $G(o) = f$, there is a neighborhood U of $o \in \boldsymbol{R}^j$ and a map $h:U \to \boldsymbol{R}^i$ such that $F(h(y))$ is equivalent to $G(y)$ for all $y \in U$.

Unfortunately, there seem to be few categories for which the program seems feasible. The important case in which it is true has L the space of C^∞ proper maps between two manifolds M and N with the equivalence relation of topological equivalence. The maps f and g are topologically equivalent if there are homeomorphisms h and k of M and N respectively so that $fh = kg$. Several attempts have been made to construct a suitable equivalence relation for the space of vector fields on a compact manifold for which the program could be carried out in even a limited sense. These attempts have met with consistent failure. For example, Shub has found an open set of topologically transitive and structurally unstable vector fields on a five-dimensional manifold [**8**]. This answers a question raised by Thom on p. 46. Takens has found drastic pathologies in the bifurcation theory of singular points of vector fields. Thus it seems that Thom has been overly optimistic about the possibilities of studying vector fields in this manner. This optimism is indicative of some of the more fundamental difficulties of the theory as Thom presents it; we shall discuss this more fully later.

There are a few additional remarks concerning Chapter 3:

(1) In discussing the stability theory of maps, Thom uses the space of C^m maps rather than the space of C^∞ maps. The stability theory for finitely

differentiable maps is in an even less satisfactory state than the theory for C^∞ maps. Note that Mather's theorem characterizing smoothly stable maps is not true in the category of C^m manifolds and maps when m is finite.

(2) It is not clear what equivalence relation is being used when Thom refers to the stability of Kolmogoroff-Moser central trajectories in Hamiltonian systems. This problem reappears later when Thom discusses weak attractors of Hamiltonian systems. There do exist central trajectories without a fundamental system of invariant neighborhoods for Hamiltonian systems having at least three degrees of freedom [**1**].

(3) On p. 47, Thom mentions the problem of Plateau. Jean Taylor has succeeded in mathematically analyzing the singularities of soap films. She has confirmed that the only singularities which arise are those which have the structure of either the central cone on the 1-skeleton of a tetrahedron or the product of an interval with the cone on three points.

(4) Moduli are important in the theory of stable maps as that theory is used by Thom. In a later paper of Thom, a singularity depending upon moduli occurs when he considers the unfoldings of a singularity with the constraint of bilateral symmetry imposed.

Chapter 4 develops a synthesis of the models presented in Chapter 1 with the theories of structural stability considered in Chapter 3. This is accomplished by assuming that the laws governing the space of internal parameters are given either by a vector field or by a potential function. The first case is the *metabolic* model; the second case is the *static* model. The local state of a regular point is then in a structurally stable attractor for the metabolic model and is a critical point of index zero of a stable function for the static model. A catastrophe for the metabolic model occurs when the local state attractor bifurcates; for the static model, a catastrophe occurs when the potential function is unstable.

Thom now invokes the structural stability hypothesis of Chapter 2. We assume that the catastrophe set of the process we are examining comes from a stable configuration of the local states. This means that the family of vector fields or functions parametrized by the space of external parameters is transverse to the bifurcation subset of the relevant function space. Consequently, we obtain geometric models as stratified sets for the catastrophe set of the process from universal unfoldings in the function space. This places restrictions on the vector fields or functions which are allowed on the space of internal parameters. For a process occurring in four dimensional space-time, they must lie in submanifolds of the bifurcation set of codimension at most 4. In the space of smooth functions on a manifold, there are only a finite number of equivalence classes of singularities of codimension at most 4. These give rise to the elementary catastrophes which are examined in Chapter 5.

Thom more or less identifies the static model with the restriction of the metabolic model to gradient dynamical systems. There is a bijection between gradient vector fields and their potential functions (normalized to be 0 at a point) but this bijection does not strictly put into correspondence the structurally stable vector fields with the stable functions. This difficulty is ignored by Thom, so let us consider the matter briefly here.

A proper function $g:N \to \boldsymbol{R}$ is stable if it satisfies the following two conditions:

(1) the critical points of g are nondegenerate, and

(2) the critical values of g are distinct.

Let $f \in C^\infty(M, \boldsymbol{R})$ be a function of finite codimension i and let $F:\boldsymbol{R}^i \to C^\infty(M, \boldsymbol{R})$ be its universal unfolding. Thom calls the set of points in $\boldsymbol{R}^i$ where condition (2) above fails to hold the *Maxwell set* of F. Thom also refers to the Maxwell set as the set of points in $\boldsymbol{R}^i$ where the absolute minimum of $F(\boldsymbol{R}^i)$ is assumed at more than one point. Depending upon the context, the Maxwell set may or may not be considered part of the catastrophe set for the process described by the unfolding. The gradients of functions in the Maxwell set will usually be stable vector fields. A structurally stable gradient vector field must satisfy condition (1) above; it must also satisfy another condition. Namely, its stable and unstable manifolds must intersect transversely. This requirement has an essentially different nature from (1) and (2) above in that it cannot be ensured by examining local properties of a function.

The nondegeneracy of critical points is the common ground between stable functions and structurally stable gradient vector fields. The degeneracy of critical points also arises in describing the singularities of wave fronts. Thom deals very briefly with this application of the theory of singularities of functions, once again leaving it to the reader to make the connection precise. We shall discuss these difficulties later. My interpretation of the mathematics underlying these situations is contained in the papers [3], [4].

As we indicated above, the geometric structure of the bifurcation set of the space of vector fields is much wilder than is the case for functions. This prompts Thom to distinguish between ordinary and essential catastrophes. Ordinary catastrophes are those points x for which, in a neighborhood of x, the catastrophe set is isomorphic to a semianalytic set without interior and having a conical structure with vertex x. Other catastrophe points are essential. Catastrophes arising from metabolic models are often essential catastrophes. Their geometric structure can be so complicated that they are literally indescribable. In this case, Thom suggests that one may be able to introduce a simpler model by performing a statistical averaging process. This *reduced* model may then have ordinary catastrophes if we have been clever in its construction. Thom considers such a reduction in the case in which the local state is given by a hamiltonian vector field. Using the invariant measure of the symplectic structure associated with the hamiltonian vector field, an entropy is defined. With the introduction of an ergodic hypothesis, the system evolves so as to maximize the entropy. This discussion is less precise than the rest of the chapter.

Chapter 5 is a description of the elementary catastrophes arising in dimensions at most four. Mathematically, it is the most straightforward part of the book. Still, it is not always easy to follow Thom's descriptions and understand his diagrams. This is particularly true when Thom considers the codimension four singularities. Godwin [2] has drawn much more careful pictures of the "parabolic umbilic". Figure 5-12b on p. 86 seems to be misdrawn.

There is a general procedure for calculating the universal unfoldings of the singularities involved in these catastrophes. One starts with a function $f: \boldsymbol{R}^n \to \boldsymbol{R}$ such that $f(o) = 0$, $df(o) = 0$, and the dimension of the vector space $m/J(f)$ is at most 4. Here m is the maximal ideal in $C^\infty(\boldsymbol{R}^n, \boldsymbol{R})$ consisting of those functions vanishing at 0 and $J(f)$ is the ideal generated by the first partial derivatives of f. There are seven different degenerate singularity types which occur, represented by the following functions with the name attached to them by Thom: x^3(fold), x^4(crisp), x^5(dove tail), x^6(butterfly), $x^3 + y^3$(hyperbolic umbilic), $x^3 - 3xy^2$ (elliptic umbilic), and $x^2y + (x^4 + y^4)/4$(parabolic umbilic). In addition there are catastrophes which arise from the transversal intersection of these catastrophes. To compute the universal unfolding of a singularity, select a set of functions $V_1, \ldots, V_k$ which project onto a basis of $M/J(f)$. The universal unfolding is then described by the map $F: \boldsymbol{R}^k \to C^\infty(\boldsymbol{R}^n, \boldsymbol{R})$ defined by $F(t_1, \ldots, t_k) = f + \sum_{i=1}^k t_i v_i$. The bifurcation set of F consists of those $t \in \boldsymbol{R}^k$ for which $F(t)$ is not stable. The Maxwell set is given by those t for which $F(t)$ has two critical points with the same critical values. The t for which $F(t)$ has a degenerate critical point are computed as those t for which the equations

$$(\partial/\partial x_i)(f + \textstyle\sum t_i v_i) = 0,$$
$$\det((\partial^2/\partial x_i\, \partial x_j)(f + \textstyle\sum t_i v_i)) = 0,$$

have a common solution in $\boldsymbol{R}^n$.

After Chapter 5, the book takes a decided turn away from mathematical fact. Rigorous argument is replaced by analogy. Chapter 6 begins with a description of various kinds of generalized catastrophes. It is impossible to give a formal description of these catastrophes. They involve such phenomena as the breaking of symmetry and the change of dimension of an attractor of a vector field (other than by Hopf bifurcation). Thom then introduces the idea of a *mean field* which plays an important role in some of the applications. In a metabolic model, it was assumed originally that the vector field was almost tangent to the space of internal parameters. Averaging the component of the vector field in the direction of the external parameters over the attractor which is the local state of the component along the direction of the internal parameters gives a vector field on the space of external parameters. This defines the mean field at the regular points of the model. The mean field is not defined at catastrophe points and usually does not have a continuous extension to the catastrophe points. Note that the mean field at a point may change if the local state of the point changes.

Next Thom gives his definitions of *morphogenetic field* and *chreod*. A *morphogenetic field* consists of a region V in space-time having a metabolic or static model without catastrophes. A *chreod* is a morphogenetic field in which time plays a privileged role: $V \subset \boldsymbol{R}^3 \times \boldsymbol{R}^+$ and there is an open set $U \subset \boldsymbol{R}^3 \times \{0\}$ such that U is contained in the closure of V and V is contained in a union of "cones of influence" with vertices in U.

Chapter 6 ends with an appendix on the structure of galaxies.

Chapter 7 begins with a cautionary note that it is speculative and that the ideas presented lack a precise mathematical formulation. Thom describes here several approaches to defining the concept of information in terms of his models. In the space $C^\infty(M, N)$ of mappings between two manifolds he gives two different definitions of the complexity of a map f. The information conveyed by f is related to its complexity. The first definition of complexity depends upon the choice of a base function $f_0 \in C^\infty(M, N)$. The complexity of f is then the minimal number of intersections of a path from f_0 to f with the bifurcation set of $C^\infty(M, N)$, the minimum being taken over paths intersecting the bifurcation set in general position. The second definition of complexity is the sum of the Betti numbers of its singular set.

In order to deal with memory and the exchange of information, it is apparently necessary to use a metabolic rather than a static model. Thom offers a suggestion of how this problem may be attacked. The interaction of two systems is represented by forming their product. The product of two stable limit cycles of a vector field gives a structurally unstable vector field on a torus. By means of random perturbations this vector field can be expected to decay into a stable one having a finite number of limit cycles. This resonance couples the two systems, transferring information between the two systems.

Chapters 8–12 present the applications of the theory to biology. Chapter 8 is quite short. It serves as a short introduction to Thom's general thoughts about biology. In Chapter 9, we come to the specific biological problems for which the theory was created. The primary goal of this chapter is to describe the local geometry of morphogenesis in animals. Thom describes the early morphogenetic movements in amphibia and birds and then compares the two. These sections of the book should be its climax. Because of the central role of these applications to the development of the theory, I sorely felt the lack of precision here and was quite disappointed. The thrust of argument in the book is that the models given here are imposed upon one by the hypothesis of stability and the type of model chosen. Yet the descriptions of morphogenesis seem ad hoc when they are finally presented. Does Thom's description of gastrulation help one organize a coherent interpretation of that process? Indeed, is there more to the theory than a language for describing discontinuities of a certain sort? These are questions that I have been unable to answer for myself.

Chapter 9 ends with an interpretation of capture and sexual reproduction in terms of the universal unfolding of the parabolic umbilic singularity. Chapter 10 is an attempt to deal with the development of an organism in global terms. The discussion is again abstract and general. A new idea is added to the structure of previous models: regulation takes place by means of a vector field defined on the section of local states. This process occurs primarily through the action of a mean field as defined in Chapter 6. An application of this sort to models for the heart beat and the propogation of a nerve impulse have been given by Zeeman [**10**]. Chapter 11 takes up intracellular processes. Chapter 12 discusses four large problems in biology: finality in biology, the irreversibility of differentiation, the origin of life, and evolution.

Chapter 13 applies the theory to language. This application has been described in more detail elsewhere [**9**]. The basic idea is that descriptions of processes in space-time can be reduced to structures arising from one and two dimensional sections of the elementary catastrophes in general position.

In his conclusion Thom once again asserts that he has not created a scientific theory but a method for creating models. He asks the question as to whether the models give experimentally verifiable predictions. The answer he gives is negative. It is only in the limited situation inside a single morphogenetic field that one could hope to construct a good quantitative model. A global understanding in quantitative terms of something as complex as an animal seems beyond human capabilities.

Is the book then just a curiosity, a metaphysical outlook on the world with which one can *do* nothing? It is possible to be optimistic about the usefulness and practicality of some of Thom's ideas. I would like to discuss here, in rather broad terms, my own impressions of the theory from a mathematical point of view and to attempt to pinpoint a few places which are attractive areas for further investigation.

The weakest link in the theory lies in the models themselves. Consider the potential function of the hyperbolic umbilic, $x^3 + y^3$, and its gradient X. If we interpret X as arising from a static model, then the singular point of X has codimension 3 in the space of potential functions. However, if we regard X as arising from a metabolic model, the bifurcation of the singular point 0 will lie in codimension 4. The reason for the distinction is that the first partial derivatives of gradient vector fields are symmetric matrices while the first partial derivatives of arbitrary vector fields are arbitrary square matrices. Thus the derivative of X at 0 is the 2×2 0-matrix and this has codimension 3 in the space of 2×2 symmetric matrices but codimension 4 in the space of 2×2 matrices.

How do we determine which theory we should use in describing the unfolding of X? The answer to the question cannot be decided on mathematical or theoretical grounds, but only by judging whether the local state dynamics are subject to the constraint of being gradient. If we are in the gradient situation, then there should be reasons for why we are. Thom is of little help in resolving this difficulty.

There is also the difficulty with the static model which we referred to earlier: the stability of a function is not equivalent to the structural stability of its gradient. Whether or not a function has two critical points with the same value is irrelevant to the structural stability of its gradient. Consequently, the role of the Maxwell set of an unfolding is ambigious in the theory of the static model. There is a choice involved as to which of several local minima is to be the local state at a point. There are different conventions for making this choice, but we are given no guidance as to which convention we should use.

Despite the central role of the models in the book, it is possible to abandon them without throwing away the entire theory. One of the key conclusions of Thom's analysis is that stable singularities play a vital role in discontinuities where they occur in the real world. It is not necessary to be limited by the models of Thom in seeking ways in which singularities enter dynamical theories. In particular, there are situations in the study of partial differential equations in which one can be quite explicit about the structure of the singularities of solutions. Thom does take note of this briefly in the book, but he hardly explores the possibilities which arise.

The archetypical situation in which the elementary catastrophes occur naturally is the theory of geometric optics. There, a light caustic is a set along which there is a discontinuity of light intensity. Generic light caustics have the local geometric form of elementary catastrophes.

There is no ambiguity here about whether or not to include the Maxwell set of an unfolding: both mathematically and physically it is irrelevant. There is no question of "competition" of attractors. Mathematically, caustics occur where functions in a family have degenerate critical points. The indices of the critical points are irrelevant; one does not care whether or not a critical point represents an attractor of the corresponding gradient vector field.

This dynamical setting for catastrophes occurs quite generally. If H is a generic first order partial differential equation, then the singularities of generic solutions of H have the local geometric structure of elementary catastrophes [**4**]. Physically, this is indeed relevant. If P is a linear partial differential operator of a quite general kind, then the characteristic equation H of P is a (nonlinear) first order differential equation. The solutions of H describe the propogation of singularities of solutions of P [**5**]. Thus the elementary catastrophes occur naturally in studying the singular parts of solutions of differential equations. From this viewpoint, it appears to be a much more reasonable task to construct quantitative models for biological catastrophes than is evident from Thom's theory. Partial differential equations can more intimately express the constraints underlying physical and biological processes than can the general models of Thom. The geometric structure of the catastrophes becomes here a part of a much more comprehensive theory.

The Maxwell set of an unfolding is connected with shock phenomena in the study of conservation laws [**6**]. The role of generic singularities in this theory is still obscure, but this looks like an attractive area of investigation. In finding solutions of conservation laws, many of the degenerate critical points of a family of functions become irrelevant. Is the geometry of discontinuity that is involved here the geometry of the Maxwell set of unfoldings?

There is another approach to making more quantitative use of the catastrophes which has been emphasized by Zeeman [**10**]. Zeeman stresses the role of surfaces of local states as themselves the underlying space for a dynamical process. His models for the heartbeat and nerve impulse are given in these terms. What is involved here is a generalized theory of relaxation oscillations. One can think of the framework by imagining there to be two time scales at work, fast time and slow time. The processes involved in fast time work on the space of internal parameters and quickly force one to approach the local state assigned to a point in the space of external parameters. With a static model at work, this gives a finite-to-one map of the manifold of possible local states onto the space of external parameters. On this manifold of possible local states, there is a vector field which may be discontinuous where the projection map onto the space of external parameters is not a local diffeomorphism. An orbit of this slow vector field may jump from one sheet of the local state manifold to another at these fold points. An analysis of this sort in one of the simplest situations leads to the van der Pol equation. A reasonable theory of this kind of discontinuous vector field awaits development. Similar problems occur in the description Smale has given for electrical circuit theory [7].

Thom emphasizes that stability can play a crucial role in determining the structural properties of a process about which we know almost nothing else. This is a remarkable idea whose influence is likely to be felt strongly in biology. Part of its power lies in the promise that one will be able to deal geometrically with situations that have been understood at best from a formal standpoint previously. There remains a gap between the models Thom puts forth in order to give mathematical content to this idea and a practical scheme for applying singularity theory to biological problems. Thom is pessimistic that this gap from singularity theory to experimental models can be bridged, but I am not so sure.

In conclusion, let me reiterate that I have largely ignored major parts of the book which deal with biology, language, and preception. Whether or not they are correct in detail, they are certainly imaginative and fascinating to read. Only René Thom could have written them. In the past, understanding Thom has been a rewarding task. There is much for each of us to discover in this marvelous book.

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REFERENCES

1. V. I. Arnold and A. Avez, *Ergodic problems in classical mechanics*, Gauthier-Villars, Paris, 1967; English transl., Benjamin, New York, 1968. MR **35** #334; **38** #1233.

2. A. N. Godwin, *Three dimensional pictures for Thom's parabolic umbilic*, Inst. Hautes Études Sci. Publ. Math. No. 40 (1971), 117–138.

3. J. Guckenheimer, *Bifurcation and catastrophe*, Proc. Internat. Sympos. in Dynamical Systems (Salvador, 1971), Academic Press, New York (to appear).

4. ———, *Catastrophes and partial differential equations*, Ann. Inst. Fourier (Grenoble) **23** (1973), fasc. 2.

5. L. Hörmander, *Fourier integral operators*, Acta Math. **127** (1971), 79–183.

6. P. Lax, *Hyperbolic systems of conservation laws*. II, Comm. Pure Appl. Math. **10** (1957), 537–566. MR **20** #176.

7. S. Smale, *On the mathematical foundations of circuit theory*, J. Differential Geometry (to appear).

8. M. Shub, M. Hirsch and C. Pugh, *Invariant manifolds* (to appear).

9. R. Thom, *Topologie et linguistique*, Essays in Topology and Related Topics (De Rham Commemorative Volume), Springer, 1971.

10. E. C. Zeeman, *Differential equations for the heartbeat and nerve impulse*, Towards a Theoretical Biology, vol. 4, C. H. Waddington (editor) (to appear).

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