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THE SPECTRA FOR OPERATORS OF A BASIC COLLECTION

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We present here the spectra of operators from a basic collection considered in the scale of Lebesgue spaces of pth power summable functions over a finite interval.

Without loss of generality we confine our attention to complex valued functions over the interval $[0, 1]$. The associated Banach spaces are denoted by L^p, $1 < p < \infty$. The results lift to any underlying bounded interval $[a, b]$ through, for example, the mappings induced by the linear map $[a, b] \ni s \mapsto (b - a)^{-1}(s - a) = t \in [0, 1]$.

The results unfold primarily through certain formal manipulations on some basic relations in an algebra of elementary operations.

On complex valued functions over the interval $[0, 1]$ we consider the operations (see [**2**])

$$\begin{aligned} J^\beta\psi(t) &= \Gamma(\beta)^{-1}\int_0^t (t - x)^{\beta - 1}\psi(x)\,dx, && 0 < \operatorname{Re}\beta, \\ &= \lim_{b \to 0+} J^{b+\beta}\psi(t) \quad (L^p\text{-limit}), && \operatorname{Re}\beta = 0, \\ &= dJ^{\beta+1}\psi(t)/dt, && -1 < \operatorname{Re}\beta < 0, \end{aligned}$$

and

$$\begin{aligned} J^{*\beta}\psi(t) &= \Gamma(\beta)^{-1}\int_t^1 (x - t)^{\beta - 1}\psi(x)\,dx, && 0 < \operatorname{Re}\beta, \\ &= \lim_{b \to 0+} J^{*b+\beta}\psi(t) \quad (L^p\text{-limit}), && \operatorname{Re}\beta = 0, \\ &= -dJ^{*\beta+1}\psi(t)/dt, && -1 < \operatorname{Re}\beta < 0. \end{aligned}$$

In addition let M^γ denote the operation given by $M^\gamma\psi(t) = t^\gamma\psi(t)$, γ-complex, and R the operation $R\psi(t) = \psi(1 - t)$. Further denote by H the finite Hilbert transform

$$H\psi(t) = \frac{1}{\pi}(\text{p.v.})\int_0^1 \frac{\psi(x)}{t - x}\,dx,$$

the integral being the Cauchy principal value. We consider also H as

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extended to a bounded operator in L^p, $1 < p < \infty$.

The algebra of elementary operations mentioned above is that generated by the collection $\{J^\beta, R, M^\gamma\}$.

The basic operators under study here are those defined by the operations

$$\psi \to J^\alpha J^{*-\alpha}\psi, \qquad |\mathrm{Re}\,\alpha| < 1. \tag{1}$$

Such operators arise quite naturally in diverse settings; see, for example, [**3**] and Kalisch [**4**].

The results to be presented here, combined with those in [**2**] and [**3**], lend themselves to the analysis of some fundamental operators; in particular, the classical operators defined for $0 < \mathrm{Re}\,\alpha < 1$ by

$$S_\alpha\psi(t) = \int_0^1 \frac{\psi(x)}{|t-x|^{1-\alpha}}\,dx, \qquad T_\alpha\psi(t) = \int_0^1 \frac{\mathrm{sgn}(t-x)}{|t-x|^{1-\alpha}}\psi(x)\,dx.$$

Note that $\Gamma(\alpha)^{-1}S_\alpha = J^\alpha + J^{*\alpha}$ and $\Gamma(\alpha)^{-1}T_\alpha = J^\alpha - J^{*\alpha}$. Then on formally factoring one encounters the operations (1). Such study will be included elsewhere.

The question of when the operations (1) give rise to bounded operators in L^p was considered in [**3**], and it is summarised here.

THEOREM 1. *The operation* $\psi \to J^\alpha J^{*-\alpha}\psi$, $|\mathrm{Re}\,\alpha| < 1$, $\alpha \neq 0$, *defines a bounded operator in* L^p, $1 < p < \infty$, *if and only if* $-1 < p\,\mathrm{Re}\,\alpha < p - 1$.

A fundamental relation in the algebra is that expressed in the following (see [**2**], [**3**]).

THEOREM 2. *For* $|\mathrm{Re}\,\beta| < 1$,

$$(\cos\pi\beta)I + (\sin\pi\beta)H = M^{-\beta}J^\beta J^{*-\beta}M^\beta. \tag{2}$$

With the aid of this basic relation (2) we are led to the following.

THEOREM 3. *For* $z = \sin\pi\alpha\cot\pi\zeta + \cos\pi\alpha$, $|\mathrm{Re}\,\alpha| < 1$, $|\mathrm{Re}\,\zeta| < 1$, $\zeta \neq 0$,

$$\begin{aligned}(M^{\alpha+\zeta}RM^{-\alpha-\zeta})&J^\alpha J^{*-\alpha}(M^{\alpha+\zeta}RM^{-\alpha-\zeta})\\ &= (\cos\pi\alpha)I + (\sin\pi\alpha)(M^{\alpha+\zeta}RM^{-\zeta})H(M^\zeta RM^{-\alpha-\zeta})\end{aligned} \tag{3}$$

and

$$\begin{aligned}(\sin\pi\zeta)^2(M^{\alpha+\zeta}RM^{-\alpha-\zeta})(zI - J^\alpha J^{*-\alpha})(M^{\alpha+\zeta}RM^{-\alpha-\zeta})(zI - J^\alpha J^{*-\alpha})\\ = (\sin\pi\alpha)^2 I.\end{aligned} \tag{4}$$

The equalities (3) and (4) are valid for $\mathrm{Re}\,\alpha \neq 0$ as arithmetical identities on applying each side to an arbitrary smooth function compactly supported in (0, 1) and in the sense of L^p $(1 < p < \infty)$ when $\operatorname{Re} \alpha = 0$.

THEOREM 4. *For* $|\operatorname{Re} a| < 1$, $|\operatorname{Re} b| < 1$ *the operation*

$$\psi \to (M^a R M^b) H (M^{-b} R M^{-a}) \psi$$

defines a bounded operator in L^p, $1 < p < \infty$, *if and only if*, $-1 < p \operatorname{Re} a < p - 1$ *and* $-1 < p \operatorname{Re} b < p - 1$.

The special case of Theorem 4 where $a = 0$ played a central role in the proof of Theorem 1; see [**3**]. In addition, the case where $\operatorname{Re}(a + b) = 0$ reduces to Theorem 3 in [**2**].

The above theorems combine to give the following.

THEOREM 5. *Let* $|\operatorname{Re} \alpha| < 1$, $\alpha \neq 0$, *and* $1 < p < \infty$. *Denote by* K_α *the operator defined in* L^p *by* $J^\alpha J^{*-\alpha}$ *where* $-1 < p \operatorname{Re} \alpha < p - 1$.

(i) *Spectrum*

$$\operatorname{sp}(K_\alpha | L^p) = \operatorname{cl} - \{\sin \pi\alpha \cot \pi\zeta + \cos \pi\alpha : |\operatorname{Re} \zeta - \tfrac{1}{2}(1 - \operatorname{Re} \alpha)| \leqq |\tfrac{1}{2}(1 - \operatorname{Re} \alpha) - p^{-1}|\}.$$

(ii) *The resolvent set*

$$\rho(K_\alpha | L^p) = \{\sin \pi\alpha \cot \pi\zeta + \cos \pi\alpha : |\operatorname{Re} \zeta + \tfrac{1}{2} \operatorname{Re} \alpha| < \tfrac{1}{2} - |\tfrac{1}{2}(1 - \operatorname{Re} \alpha) - p^{-1}|, \zeta \neq 0\}$$

and for $z \in \rho(K_\alpha | L^p)$ *the resolvent*

$$(zI - K_\alpha)^{-1} = (\csc \pi\alpha)^2 (\sin \pi\zeta)^2 (M^{\alpha+\zeta} R M^{-\alpha-\zeta})(zI - K_\alpha)(M^{\alpha+\zeta} R M^{-\alpha-\zeta})$$

where $z = \sin \pi\alpha \cot \pi\zeta + \cos \pi\alpha$.

REMARKS. (I) In the case $p(1 - \operatorname{Re} \alpha) = 2$ the spectrum is the circular arc with endpoints $e^{\pm i\pi\alpha}$ that contains the point 1. The limiting, or degenerate, case where $\operatorname{Re} \alpha = 0$ is an interval on the real axis.

(II) For a case where $p = 2$ the spectrum is the segment of the disk bounded by the circle through the points 0 and $e^{\pm i\pi\alpha}$ that is cut off by the secant with endpoints $e^{\pm i\pi\alpha}$ and that contains the point 1. As in (I) the limiting case where $\operatorname{Re} \alpha = 0$ is again the interval on the real axis.

(III) The cases where $\alpha = i\tau$ (that is, $\operatorname{Re} \alpha = 0$) relate to the continuous boundary group of the holomorphic semigroup (see Hille–Phillips [**1**, §23.16]). Here the spectrum for each τ is described using the disks bounded by the circles centered at c and $\bar{c}$,

$$c = \cosh \pi\tau + i(\sinh \pi|\tau|)|\cot 2\pi/p|,$$

that pass through the points $e^{\pm\pi|\tau|}$.

(a) For $0 < |p^{-1} - \tfrac{1}{2}| \leqq \tfrac{1}{4}$ the spectrum is the intersection of the disks.

(b) For $p = 2$ the spectrum is the interval $[e^{-\pi|\tau|}, e^{\pi|\tau|}]$.

(c) For $|p^{-1} - \frac{1}{2}| \geqq \frac{1}{4}$ the spectrum is the union of the disks.

This is similar to the situation for the finite Hilbert transform given in [**2**].

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