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PERIODIC SOLUTIONS OF AUTONOMOUS FUNCTIONAL DIFFERENTIAL EQUATIONS

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ABSTRACT. The purpose of this note is to indicate some applications of a new fixed point theorem to the question of periodic solutions of nonlinear autonomous functional differential equations. The techniques developed give the standard periodicity examples in the literature and some new results, notably for the neutral case, which do not seem accessible by previous methods.

1. If X is a Banach space and A is a bounded subset of X, define $\gamma(A)$, the measure of noncompactness of A, to be $\inf\{d > 0 : A$ has a finite covering by sets of diameter less than $d\}$. This is a notion due to C. Kuratowski [**13**]. G. Darbo observed [**4**] that if $\overline{\text{co}}(A)$ denotes the convex closure of a set A and if $A + B = \{a + b : a \in A, b \in B\}$ for sets A and B, then (1) $\gamma(\overline{\text{co}}(A)) = \gamma(A)$ and (2) $\gamma(A + B) \leqq \gamma(A) + \gamma(B)$. It is trivially true that (3) $\gamma(A \cup B) = \max\{\gamma(A), \gamma(B)\}$.

For applications it is sometimes convenient to generalize the above idea slightly. If μ is a function which assigns to each bounded subset A of X a real number $\mu(A)$, we say that μ is a generalized measure of noncompactness if μ satisfies properties (1), (2) and (3) above and if there exist positive constants m and M such that $m\mu(A) \leqq \gamma(A) \leqq M\mu(A)$ for every set $A \subset X$. If J is a closed bounded interval of $\boldsymbol{R}$ and $C^1(J, \boldsymbol{R}^n)$ denotes the Banach space of continuously differentiable maps from J to $\boldsymbol{R}^n$ with any of the standard norms, then if

$$\mu(A) = \lim_{\delta \to 0; \delta > 0} (\sup\{|x'(t) - x'(s)| : x \in A, t, s \in J, |t - s| < \delta\}),$$

μ is an example of a generalized measure of noncompactness on $C^1(J, \boldsymbol{R}^n)$.

If U is a subset of a Banach space X, $f : U \to X$ is a continuous map, and μ is a generalized measure of noncompactness, then we shall say that f is a k-set-contraction with respect to μ if for every bounded set $A \subset U$, $f(A)$ is bounded and $\mu(f(A)) \leqq k\mu(A)$. If G is a closed, convex subset of X, U is a bounded open subset of G, and $f : \overline{U} \to G$ is a k-set-contraction with respect to μ, $k < 1$, then if $f(x) \neq x$ for $x \in \overline{U} - U$, there is an integer defined, called the fixed point index of f on U and written $i_G(f, U)$. Details are given in [**15**], where the fixed point index is actually defined for a larger class of maps which are defined on open subsets of certain metric

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ANR's. The fixed point index can be thought of as an algebraic count of the number of fixed points of f in U. If $G = X$ and f is compact, $i_G(f, U) = \deg(I - f, U, 0)$, the Leray-Schauder degree of $I - f$ on U.

Before stating our main fixed point theorem we need one more definition. Suppose that C is a topological space, $x_0 \in C$, W is an open neighborhood of x_0, and $f: W - \{x_0\} \to C$ is a continuous map. Then we shall say that x_0 is an "ejective point of f" if there exists an open neighborhood U of x_0 such that for every $x \in U - \{x_0\}$, there exists an integer $m = m(x)$ such that $f^m(x)$ is defined and $f^m(x) \notin U$. This definition is a slight generalization of one given by F. E. Browder in [**2**], where he assumed that f is defined and continuous at x_0 and $f(x_0) = x_0$. However, for some applications f cannot be defined continuously at x_0.

THEOREM 1. *Let G be a closed, bounded, convex and infinite dimensional subset of a Banach space X, $x_0 \in G$, and $f: G - \{x_0\} \to G$ a continuous map such that x_0 is an ejective point of f. Assume that f is a k-set-contraction with respect to μ, $k < 1$, where μ is a generalized measure of noncompactness on X. Then if U is an open neighborhood of x_0 in G such that f has no fixed points in $\bar{U} - \{x_0\}$, $i_G(f, G - \bar{U}) = 1$ and f has a fixed point in $G - \bar{U}$. If G is finite dimensional (not equal to a point) and x_0 is an extreme point of G, the same conclusion holds.*

Theorem 1 is not hard to obtain by use of some ideas from [**2**] and standard techniques in asymptotic fixed point theory. Its significance lies in its direct applicability to a number of concrete analytic problems. We should mention, however, that there are situations (see Theorem 3 below) in which Theorem 1 seems inapplicable and other asymptotic fixed point theorems are needed.

COROLLARY 1. *Let G be a closed, bounded, convex and infinite dimensional subset of a Banach space X, μ a generalized measure of noncompactness on X and $f: G \to G$ a k-set-contraction with respect to μ, $k < 1$. Then f has a fixed point which is not an ejective point of f. If x_0 is an ejective fixed point of f and U is an open neighborhood of x_0 in G such that $f(x) \neq x$ for $x \in \bar{U} - \{x_0\}$, then $i_G(f, U) = 0$.*

Corollary 1 is a direct generalization of Browder's work in [**1**], [**2**] and Jones's fixed point theorems in [**11**].

COROLLARY 2. *Let G be a closed, convex and infinite dimensional subset of a Banach space X such that $0 \in G$. If μ is a generalized measure of noncompactness on X, $U = \{x \in G: \|x\| < R\}$, and $f: \bar{U} - \{0\} \to G$ is a k-set-contraction with respect to μ, $k < 1$, such that 0 is an ejective point of f and $f(x) \neq tx$ for $t \geqq 1$ and $\|x\| = R$, then if $W = \{x \in G: r < \|x\| < R\}$ and $f(x) \neq x$ for $0 < \|x\| \leqq r$, $i_G(f, W) = 1$ and f has a fixed point in W.*

2. Using Theorem 1 and its corollaries we have obtained the periodicity results given by G. S. Jones in [**9**], [**11**] and R. B. Grafton in [**5**], [**6**]. Rather than repeat these results we mention some simple new examples which appear inaccessible by previous methods. All of the examples given below admit more or less mechanical generalization, but for ease of exposition we restrict ourselves to the cases below. In all of the examples we give, the fixed point index of an appropriately defined map equals one; in a future paper we hope to give examples of a completely different type for which a fixed point index equals minus one.

Our first example is a generalized van der Pol equation of the type considered by Grafton in [**5**], [**6**]. The case $k < 0$ below partially answers a question raised by Grafton in [**6**].

THEOREM 2. *The equation*

$$x''(t) - \varepsilon x'(t)[1 - x^2(t)] + x(t - r) - kx(t) = 0$$

has a nontrivial periodic solution of period greater than $2r$ *if* $\varepsilon > 0$, $r > 0$ *and* $-k_0 < k < 1$, *where* $k_0 = \min(\varepsilon/r, 2/r^2)$.

The following equation was mentioned by Halanay and Yorke in [**7**]. Its analysis involves some nonstandard techniques.

THEOREM 3. *The equation* $x'(t) = \alpha x(t - 1 - |x(t)|)(1 - x^2(t))$ *has a nontrivial periodic solution of period greater than* 2 *if* $\alpha > \pi/2$.

THEOREM 4. *The neutral functional differential equation*

$$x'(t) = [-\alpha x(t - 1) + (k/m + 1)(d/dt)x^{m+1}(t - 1)][1 - x^2(t)]$$

has a nontrivial periodic solution of period greater than 2 *if* $\alpha > \pi/2$, $m \geqq 1$ *and* $|k| \leqq [(m + 1)/4][1 + 2/m - 1]^{m-1/2}$ ($|k| \leqq \frac{1}{2}$ if $m = 1$).

Since the equation $x'(t) = -\alpha x(t - 1)(1 - x^2(t))$ is perhaps the best understood of all nonlinear autonomous FDE's (with regard to periodic solutions), the neutral FDE in Theorem 4 is one of the simplest nonlinear, neutral FDE's imaginable, but even in this case there are unanswered questions and Theorem 4 is not best possible.

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