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ON THE DIFFERENTIALS IN THE LYNDON-HOCHSCHILD-SERRE SPECTRAL SEQUENCE

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In this announcement we will state some results on the torsion of the differentials in the Lyndon-Hochschild-Serre (L-H-S) spectral sequence in the homology theory of groups and give some applications. Detailed proofs and further applications will appear elsewhere.

1. **Main result.** Let

$$N \rightarrowtail G \twoheadrightarrow Q \tag{1.1}$$

be a group extension with N abelian, characterized by $\alpha \in H^2(Q;N)$, and let A be a G-module. Then there is a L-H-S spectral sequence (see [**5**]) $\{E_r^{mq}(A), d_r^\alpha\}$, associated with (1.1), with $E_2^{mq}(A) = H_m(Q;H_q(N;A))$, converging to the homology of G with coefficients in A.

To the authors' knowledge, only the differential d_2^α has been studied ([**1**], [**2**], [**3**], [**4**]); nothing seems to be known about the higher differentials $d_r^\alpha, r \geqq 3$.

To state our main result we introduce certain numerical functions κ, λ, σ. For any natural number h and any prime p, we write $p^e \| h$ to mean that $p^e | h$ but $p^{e+1} \nmid h$. Let q, f, n be natural numbers and define $a(p), b(p)$ by

$$p^{a(p)} \| f, \qquad b(p) = \min(q, a(p) + 1).$$

Let n admit the prime-power factorization $n = p_1^{s_1} p_2^{s_2} \cdots p_l^{s_l}$, and define the functions κ, λ, σ by

$$\kappa(f, n) = \prod_{i=1}^{l} p_i^{s_i + a(p_i)},$$

$$\lambda(q, f, n) = \prod_{(p-1)|f;\, p \neq p_1, p_2, \ldots, p_l} p^{b(p)},$$

$$\sigma(q, f, n) = \begin{cases} 2\kappa\lambda & \text{if } f \text{ is even and } 2\|n \text{ or if } f \text{ is even, } n \text{ is odd and } a(2) + 2 \leqq q, \\ \kappa\lambda & \text{otherwise.} \end{cases} \tag{1.2}$$

Our main result is

THEOREM 1.1. *Let* (1.1) *be characterized by* $\alpha \in H^2(Q;N)$ *of order* n. *Then, provided that either*

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(a) *$q + r \leqq 3$ and A is a Q-module which is torsion-free as abelian group, or*

(b) *N is torsion-free and A is an arbitrary Q-module, we have*

$$\sigma(q, r - 1, n)d_r^\alpha = 0, \quad r \geqq 2,$$

where

$$d_r^\alpha : E_r^{mq}(A) \to E_r^{m-r,q+r-1}(A).$$

We note that condition (a) above covers the portion of the spectral sequence giving information on the Schur multiplicator of G.

SKETCH OF PROOF. Let $f_k : N \to N$ denote multiplication by $kn + 1$. Since $f_k(\alpha) = (kn + 1)\alpha = \alpha \in H^2(Q; N)$, it follows that there is a map $\phi_k : G \to G$ giving rise to a commutative diagram.

$$\text{(1.3)} \qquad \begin{array}{ccccc} N & \rightarrowtail & G & \twoheadrightarrow & Q \\ \downarrow{\scriptstyle f_k} & & \downarrow{\scriptstyle \phi_k} & & \| \\ N & \rightarrowtail & G & \twoheadrightarrow & Q \end{array}$$

Then (1.3) yields in its turn a commutative square.

$$\text{(1.4)} \qquad \begin{array}{ccc} E_r^{mq}(A) & \xrightarrow{d_r^\alpha} & E_r^{m-r,q+r-1}(A) \\ {\scriptstyle f_{k*}}\downarrow & & {\scriptstyle f_{k*}}\downarrow \\ E_r^{mq}(A) & \xrightarrow{d_r^\alpha} & E_r^{m-r,q+r-1}(A) \end{array}$$

Under the hypotheses of the theorem one may identify $(f_k)_* : E_r^{mq}(A) \to E_r^{mq}(A)$ with multiplication by $(kn + 1)^q$, so that (1.4) leads to the relation

$$\text{(1.5)} \qquad \theta(k)d_r^\alpha = 0,$$

where

$$\text{(1.6)} \qquad \theta(k) = (kn + 1)^q((kn + 1)^{r-1} - 1).$$

Since the gcd of the integers $\theta(k)$ (1.6) is precisely $\sigma(q, r - 1, n)$, the theorem follows from (1.5).

As a special case, one may consider the split extension (semidirect product) $N \rightarrowtail N \rtimes Q \twoheadrightarrow Q$. Then, under the hypotheses of Theorem 1.1, we find

$$d_r = 0 \quad \text{if } q = 0,$$

$$2d_r = 0 \quad \text{if } r \text{ is even and } q \geqq 1,$$

$$\lambda d_r = 0 \quad \text{if } r \text{ is odd and } 1 \leqq q \leqq a(2) + 1,$$

$$2\lambda d_r = 0 \quad \text{if } r \text{ is odd and } a(2) + 2 \leqq q.$$

Here $\lambda = \prod_{(p-1)|(r-1)} p^{b(p)}$.

REMARKS. (a) A similar analysis of the torsion of d_r^α may be made in case N is the direct product of two cyclic groups.

(b) The estimates for the torsion of d_r^α may be sharpened when the extension (1.1) is central. One then obtains, under the hypotheses of Theorem 1.1.

$$\kappa(r-1, n)d_r^\alpha = 0, \quad \text{unless } 2\|n, r \text{ odd},$$

$$2\kappa(r-1, n)d_r^\alpha = 0, \quad \text{if } 2\|n, r \text{ odd}.$$

2. **Applications.** (i) *The Schur multiplicator of a semidirect product* (see also [**4**]).

PROPOSITION 2.1. *Let* $N \rightarrowtail N \rtimes Q \twoheadrightarrow Q$ *be the split extension. Suppose that either*

(a) $2: N \to N$ *is an automorphism* (*e.g.*, N *is torsion without* 2-*torsion*), *or*

(b) Q *is of odd order.*

Then $H_2(N \rtimes Q) = R_2 \oplus H_2 Q$, *where there is a short exact sequence*

$$(2.1) \qquad (H_2 N)_Q \rightarrowtail R_2 \twoheadrightarrow H_1(Q; N).$$

The sequence (2.1) *splits under hypothesis* (a) (*see* [**4**]).

(ii) *The order of the Schur multiplicator of a finite extension* (1.1). We again suppose $\alpha \in H^2(Q; N)$, characterizing (1.1), to be of order n. Let π be the set of primes dividing $2n$ and let π' be the complementary set of primes. For any natural number h, let $\pi(h)$ be the π-primary factor of h and $\pi'(h)$ the π'-primary factor of h.

PROPOSITION 2.2. (a) $\pi'|H_2 G| = \pi'|H_2 Q| \cdot \pi'|H_1(Q; N)| \cdot \pi'|(H_2 N)_Q|$.

(b) $\pi|nH_2 Q| \cdot \pi|nH_1(Q; N)| \cdot \pi|t(H_2 N)_Q| \leqq \pi|H_2 G| \leqq \pi|H_2 Q| \cdot \pi|H_1(Q; N)|$ $\pi|(H_2 N)_Q|$, *where* $t = 2n^2$ *if* n *is odd or* $4|n$, *and* $t = 4n^2$ *if* $2\|n$.

(iii) *The rank of* $H_n G$ *for a finite extension* $N \rightarrowtail G \twoheadrightarrow Q$. We again refer to (1.1), characterized by $\alpha \in H^2(Q; N)$ of finite order. With no further assumption on N we infer

PROPOSITION 2.3. rank $H_n G = \sum_{m=0}^{n}$ rank $H_m(Q; H_{n-m} N)$.

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