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BASIS GRAPHS OF PREGEOMETRIES

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A **combinatorial pregeometry**, or **matroid**, may be defined as a finite set of **elements** E and a collection of **bases** $\mathscr{B}$, all subsets of E, such that for all $B, B' \in \mathscr{B}$ and any $e' \in B' - B$, there exists $e \in B - B'$ for which $B - e + e' \in \mathscr{B}$. This exchange axiom suggests it is fruitful to represent a pregeometry $\mathscr{M}$ by a graph: Let there be a vertex for each basis and an edge for each pair of bases differing by a single exchange. We get the **basis graph** $BG(\mathscr{M})$. A special case of this construct, tree graphs, has been studied for several years [**3**]. The more general situation has attracted attention only recently [**1**], [**4**].

Our purpose here is to announce our own studies of pregeometry basis graphs [**6**], [**7**], and to state some of our key findings. We have recently learned that some of our results and methods are similar to those discovered about the same time by Cunningham [**2**] and also by Holzmann, Norton and Tobey [**5**]. In particular, Theorems 2 and 3 below are in this category.

We first characterize basis graphs. Given any graph $G(\mathscr{V}, \mathscr{E})$, suppose $\delta(v', v'') = 2$ and $\mathscr{V}' \subset \mathscr{V}$ consists of v', v'' and all vertices adjacent to both. Then the induced subgraph $\langle \mathscr{V}' \rangle$ is called the **common neighbor subgraph** $CN(v', v'')$, or simply a CN. In a basis graph each CN is either a square (4-cycle), a pyramid (with square base), or an octahedron. Again in any graph, a **leveling** from v_0 is a partition of $\mathscr{V}$ into

$$\mathscr{V}_k = \{v : \delta(v, v_0) = k\}, \qquad k = 0, 1, \ldots.$$

In any leveling of a basis graph, each octahedral CN lies in one of three positions: (i) all in one level; (ii) across two levels, three adjacent vertices in each; or (iii) across three levels, one vertex in the highest, one not adjacent to it in the lowest, and four in between. Any other CN must lie as would an induced subgraph of such an octahedron. We call this the **positioning condition**. Finally, the **neighborhood subgraph** $N(v)$ is the induced subgraph on the vertices adjacent to v (v not included).

THEOREM 1. *G is a basis graph iff*

(1) *it is connected*;

(2) *each CN is a square, pyramid or octahedron*;

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(3) *in every leveling, every CN meets the positioning condition*; *and*
(4) *for some* v_0, $N(v_0)$ *is the line graph of a bipartite graph.*

There is an alternate form of this characterization in which the positioning condition is assumed for the leveling from v_0 only, but another condition forbidding two particular induced subgraphs is added. We can also show that (4) is redundant in many cases. We suspect it is redundant in all. Indeed, we conjecture that (1) and (2) alone suffice if there are no square CN's; unfortunately, they do not suffice otherwise.

The proof of Theorem 1 leads quickly to two other results. First, there is a simple characterization of basis graphs in terms of mappings into the small subclass whose CN's are all octahedra. Second, if two paths differing by a single vertex are considered homotopic, all basis graphs are homotopically trivial. More generally, this homotopy relates to graph products in the same way ordinary homotopy relates to topological products, i.e.,

$$\pi(G_1 \times G_2) \approx \pi(G_1) \times \pi(G_2).$$

We characterize pregeometries whose basis graphs contain only one or two types of CN's. The most interesting of these results is that $\mathscr{M}$ *is binary* (see [**9**] or [**11**]) *iff* $BG(\mathscr{M})$ *contains no induced octahedra.*

Given $G(\mathscr{V}, \mathscr{E})$, $\langle \mathscr{V}' \rangle$ is **shortest path complete** (SPC) if whenever $v \in \mathscr{V}$ is on a shortest path in G between $v', v'' \in \mathscr{V}'$, then $v \in \mathscr{V}'$. Tutte [**9**] has characterized certain important classes of pregeometries in terms of forbidden minors. We show that minors of $\mathscr{M}$ correspond to SPC's of $BG(\mathscr{M})$. This allows us to find some analogues of his theorems. For instance, $\mathscr{M}$ *is regular iff no SPC of* $BG(\mathscr{M})$ *is an octahedron or a certain graph with* 29 *vertices.* Planar-graph pregeometries can be characterized by further requiring that no SPC be the tree graph of K_5 or $K_{3,3}$. However, graphic pregeometries cannot be characterized in this way, for a pregeometry and its dual have isomorphic basis graphs.

For any $\mathscr{M}(E, \mathscr{B})$ we may assign to each $B \in \mathscr{B}$ a 0-1 **circuit matrix** $C(B)$ with rows indexed by B, columns by $D = E - B$, and a 1 in entry (b, d) iff $B - b + d \in \mathscr{B}$. For graphic pregeometries these are closely related to the usual cycle and cocycle matrices. More generally, for each binary pregeometry, one can get from any circuit matrix to any other by the standard pivoting rules of linear programming (applied to the field F_2). In fact, $\{C(B) : B \in \mathscr{B}\}$ is just a combivalence class as defined by Tucker [**8**]. For arbitrary pregeometries, one may still pivot between circuit matrices; with just one exception the rules are the same as for a combivalence class. We use this result, first demonstrated by Yoseloff [**10**], to obtain simply several propositions. Among these are the next theorem and the results already mentioned on basis graphs with restricted CN's.

$\mathscr{M}(E, \mathscr{B})$ is **proper** if no element is in every basis or outside of every one. Improper elements do not affect $BG(\mathscr{M})$. Also, if there exist $\mathscr{M}_i(E_i, \mathscr{B}_i)$, $i = 1, 2$, where the E_i are nonempty and partition E, and $B = \{B_1 \cup B_2 : B_i \in \mathscr{B}_i\}$, then $\mathscr{M}$ is a nontrivial **sum**.

THEOREM 2. *If $\mathscr{M}$ is proper, the following are equivalent:*
(1) *$\mathscr{M}$ is a nontrivial sum;*
(2) *$BG(\mathscr{M})$ is a nontrivial product;*
(3) *for some B, $N(B)$ in $BG(\mathscr{M})$ is disconnected.*

We also use circuit matrices to study **polars**, the tops and bottoms of levelings. Not only is every polar itself a basis graph, it is even the basis graph of a sum. This is trivially true for a top polar, since it is a single vertex. However, we generalize the notion of leveling in a way which untrivializes this fact, while at the same time bringing out an up-down symmetry.

Finally, we have noted that $\mathscr{M}$ and its dual $\mathscr{M}^*$ have isomorphic basis graphs. There is a partial converse. We write $\mathscr{M} \approx \mathscr{M}'$ if there is a bijection of elements which preserves bases.

THEOREM 3. *If $\mathscr{M}, \mathscr{M}'$ are proper and $BG(\mathscr{M}) \approx BG(\mathscr{M}')$, then there exist $\mathscr{M}_1, \mathscr{M}_2$ where*

$$\mathscr{M} = \mathscr{M}_1 + \mathscr{M}_2, \qquad \mathscr{M}' \approx \mathscr{M}_1 + \mathscr{M}_2^*.$$

REMARK. In many situations it is natural to restrict attention from pregeometries to geometries; see Crapo and Rota [**11**]. However, this is usually not the case when one works with bases, where the restriction is that each pair of elements must appear in some basis. In particular, there does not seem to be a nice refinement of Theorem 1 which characterizes basis graphs of geometries.

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