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BOUNDARY BEHAVIOR OF THE CARATHÉODORY, KOBAYASHI, AND BERGMAN METRICS ON STRONGLY PSEUDOCONVEX DOMAINS IN C^n WITH SMOOTH BOUNDARY

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Let G be a bounded domain in C^n. Let Δ be the unit disk in C. Let $\Delta(G)$ be the set of holomorphic mappings from G to Δ, and $G(\Delta)$ the set of holomorphic mappings from Δ to G. The Carathéodory metric on G (i.e. the infinitesimal form, as in [7] is defined by

$$F_C(z, \xi) = \sup_{f \in \Delta(G)} |f_*(\xi)| = \sup_{f \in \Delta(G)} \left| \sum_{i=1}^{n} \frac{\partial f}{\partial z_i}(z)\xi_i \right|, \qquad z \in G, \xi \in C^n.$$

The Kobayashi metric on G (infinitesimal form) is defined by [**8**] $F_K(z, \xi) = \inf\{\alpha | \exists f \in G(\Delta) \text{ with } f(0) = z, \ f'(0) = \xi/\alpha, \ \alpha > 0\}$. For the definition of the Bergman metric see [**1**] or [**4**]. We take

$$F_B(z, \xi) = (ds^2(z, \xi))^{\frac{1}{2}}$$

in the notation of [**4**].

We consider the boundary behavior of these metrics for fixed ξ. The notable features are (i) the different limiting behavior in tangential and normal directions (cf. Stein [**9**]), and (ii) the appearance of the Levi form as the limiting value of a quantity defined inside the domain.

THEOREM. *Let G be a (bounded) strongly pseudoconvex domain in C^n with C^2 boundary. Let $z_0 \in \partial G$. Let φ be a C^2 defining function for ∂G such that $|\nabla_z \varphi(z_0)| = 1$. Let $F(z, \xi)$ be either the Carathéodory or the Kobayashi metric on G. Then*

$$\lim_{z \to z_0} F(z, \xi) d(z, \partial G) = \tfrac{1}{2} |\nabla_z \varphi(z_0) \cdot \xi| = \tfrac{1}{2} |\xi_N(z_0)|.$$

If $\nabla_z \varphi(z_0) \cdot \xi = 0$, i.e. ξ is a tangent vector to ∂G at z_0, then

$$\lim_{z \to z_0; z \in \Lambda} (F(z, \xi))^2 d(z, \partial G) = \tfrac{1}{2} \mathscr{L}_{\varphi, z_0}(\xi) = \tfrac{1}{2} \sum_{i,j=1}^{n} \frac{\partial^2 \varphi}{\partial z_i \partial \bar{z}_j}(z_0) \xi_i \bar{\xi}_j.$$

$d(z, \partial G)$ is the Euclidean distance to the boundary. $\nabla_z \varphi$ is the vector $(\partial\varphi/\partial z_1, \ldots, \partial\varphi/\partial z_n)$, and $\nabla_z \varphi(z_0) \cdot \xi = \sum_{i=1}^{n} (\partial\varphi/\partial z_i)(z_0)\xi_i = \xi_N(z_0)$ is the (complex) normal component of ξ at z_0. Λ in the second limit denotes a cone of arbitrary aperture with vertex at z_0 and axis the interior normal to ∂G.

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For the Bergman metric a factor of $(n + 1)^{1/2}$ appears in the first limit, and $(n + 1)$ in the second. The first result was obtained by Diederich in [**4**] with the restriction $z \in \Lambda$. His methods easily give the second.

Full proofs will appear in the author's thesis and will be published elsewhere. E. Stein has informed me that some related results for the Carathéodory metric have been obtained by G. Henkin [**10**].

SKETCH OF PROOF. As in [**1**], [**4**], and [**5**], we introduce suitable domains of comparison which pass through z_0 and in a sufficiently small neighborhood of z_0 lie inside or outside G. Explicit calculations are made for these domains. The reduction to local questions is made possible by (i) the monotonicity property of $F(z, \xi)$

$$G \subset G' \Rightarrow F_G(z, \xi) \geqq F_{G'}(z, \xi), \qquad z \in G;$$

(ii) (a) for the Carathéodory metric, an approximation theorem for bounded holomorphic functions due essentially to Diederich (Theorem 1 in [**4**]). Together with the introduction of a peak function at z_0, this gives

PROPOSITION. *Let φ, the function defining ∂G, be strictly pluri-subharmonic in a neighborhood V of $z_0 \in \partial G$. Let $\varepsilon > 0$, and let $G_1 = \{z \in V | \varphi(z) - \varepsilon\|z - z_0\|^2 < 0\}$. Then*

$$\overline{\lim_{z \to z_0}} \frac{F_{G_1}(z, \xi)}{F_G(z, \xi)} \leqq 1.$$

(b) For the Kobayashi metric, an estimate of Royden (Lemma 2 in [**8**]) which easily gives

PROPOSITION. *Let G be a strongly pseudoconvex domain in $\mathbf{C}^n$. Let V be a neighborhood of $z_0 \in \partial G$. Then*

$$\lim_{z \to z_0} \frac{F_{G \cap V}(z, \xi)}{F_G(z, \xi)} = 1.$$

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