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A RESTRICTION ON THE PARAMETERS OF A SUBQUADRANGLE

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1. **Introduction.** A *generalized quadrangle of order* (s, t) is a finite incidence plane P with $v_1 = (1 + t)(1 + st)$ lines, $v_2 = (1 + s)(1 + st)$ points, and a symmetric incidence relation I satisfying the following axioms (cf. [**1**]):

I-1. No two lines of P are incident with two points in common.

I-2. If x is a point of P and L is a line of P such that $x \not\mathrm{I}\, L$ (i.e., x is not incident with L), then there is a unique pair (x', L') consisting of a point and line, respectively, such that $x \mathrm{I} L'$, $x' \mathrm{I} L'$, and $x' \mathrm{I} L$.

I-3. Each line (point) is incident with $1 + s$ points ($1 + t$ lines).

Throughout this note, P will denote a generalized quadrangle of order (s, t) and Q a subquadrangle of P of order (s', t') with $1 \leqq s' \leqq s$, $1 \leqq t' \leqq t$. In [**4**], Thas gives a number of restrictions on s' and t' in terms of s and t. Two of them are as follows in case $s' < s$ and $t' < t$:

(1) $s'(t')^2 < st$ and $t'(s')^2 < st$.

(2) If $t = s$ and $t' = s' \geqq 13$, then $s^2 > 3(s')^3$.

It is the purpose of this note to give the following improvement of (1) and (2), which is a "best possible" result in the sense that, for each prime power s', the case $s = (s')^2$ ($s = t$, $s' = t'$) does arise.

THEOREM. *With* s, s', t, t' *as above, it must be that*

$$\text{(a) } s \geqq s't' \text{ or } s = s' \quad \textit{and dually} \quad \text{(b) } t \geqq s't' \text{ or } t = t'.$$

Thas examines rather thoroughly the case $s = s'$, $t > t'$, and we refer the reader to [**4**] for several results in this case.

2. **Proof of the Theorem.** Our proof of the Theorem is based on ideas of D. G. Higman and C. Sims, particularly as developed in [**2**] and [**3**].

Let G be the graph whose vertices are the points of P and whose edges are the pairs of noncollinear points of P. Let A be the (0,1) adjacency matrix of G defined in terms of some fixed ordering of the vertices of G. Then A is symmetric with characteristic roots $-s$, t, and s^2t. Partition the vertices of G into two sets Δ_1 and Δ_2 as follows: Δ_1 is the set of points of Q; Δ_2 is the set of points of P not in Q. For convenience put

$$n_1 = |\Delta_1| = (1 + s')(1 + s't'), \qquad n_2 = |\Delta_2| = (1 + s)(1 + st) - n_1.$$

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For $1 \leqq i, j \leqq 2$, let e_{ij} be the number of ordered pairs (x, y) for which $x \in \Delta_i$, $y \in \Delta_j$, and (x, y) is an edge of G. Let $\hat{A}$ be the 2×2 matrix whose (i, j) entry is e_{ij}/n_i. According to the theorem of Sims quoted on p. 144 of [**2**], the characteristic roots of $\hat{A}$ must lie between $-s$ and s^2t.

A straightforward computation shows that

$$\hat{A} = \begin{pmatrix} (s')^2t' & s^2t - (s')^2t' \\ \dfrac{n_1(s^2t - (s')^2t')}{n_2} & s^2t - \dfrac{n_1(s^2t - (s')^2t')}{n_2} \end{pmatrix}. \tag{3}$$

It is easy to see that s^2t is a root of $\hat{A}$, so that $\mathrm{tr}(\hat{A}) - s^2t = (s')^2t' - (n_1(s^2t - (s')^2t'))/n_2 = r$ is the other root. The condition that $-s \leqq r$ is then easily shown to be equivalent to

$$0 \leqq (s - s't')(s - s'). \quad \square \tag{4}$$

COROLLARY. *If a generalized quadrangle P of order s (i.e., $s = t$) has a subquadrangle Q of order s', then $s \geqq (s')^2$.*

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