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DEGREE THEORY FOR NONCOMPACT MULTIVALUED VECTOR FIELDS

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Introduction. In this note we indicate the development and state the properties of a degree theory for a rather general class of multivalued mappings, the so-called ultimately compact vector fields, and then use this degree to obtain fixed point theorems. As will be seen, these results unite and extend the degree theory for single-valued ultimately compact vector fields in [**13**] and the degree theory for multivalued compact vector fields in ([**5**], [**8**]) and also serve to extend to multivalued mappings the fixed point theorems for single-valued mappings obtained in [**1**], [**2**], [**3**], [**9**], [**10**], [**13**], and others (see [**13**]) and to more general multivalued mappings the fixed point theorems in [**4**], [**6**], [**8**]. The detailed proofs of the results presented in this note will be published elsewhere.

1. Let X be a metrizable locally convex topological vector space. If $D \subset X$ we denote by $K(D)$ and $CK(D)$ the family of closed convex, and the family of compact convex subsets of D, respectively. We also use $\bar{D}$ (or cl D), ∂D, and clco D to denote the closure, boundary and convex closure of D, respectively. To define what we mean when we say that the upper semicontinuous (u.s.c.) mapping $T: D \to K(X)$ is ultimately compact, we employ a construction of a certain transfinite sequence $\{K_\alpha\}$ utilized by Sadovsky [**13**] in his development of the index theory for ultimately compact single-valued vector fields. Let $K_0 = \text{clco } T(D)$, where $T(A) = \bigcup_{x \in A} T(x)$ for $A \subset D$. Let η be an ordinal such that K_β is defined for $\beta < \eta$. If η is of the first kind we let $K_\eta = \text{clco } T(D \cap K_{\eta-1})$, and if η is of the second kind we let $K_\eta = \bigcap_{\beta<\eta} K_\beta$. Then $\langle K_\alpha \rangle$ is well defined and such that $K_\alpha \subset K_\beta$ if $\alpha > \beta$. Consequently, there exists an ordinal γ such that $K_\beta = K_\gamma$ if $\beta \geqq \gamma$. We define $K = K(T, D) = K_\gamma$ and observe that clco $T(K \cap D) = K$. The mapping T is called *ultimately compact* if either $K \cap D = \varnothing$ or if $T(K \cap D)$ is relatively compact.

DEFINITION 1. Let $D \subset X$ be open with $T: \bar{D} \to K(X)$ ultimately compact and such that $x \notin T(x)$ if $x \in \partial D$. If $K(T, \bar{D}) \cap D = \varnothing$ we define $\deg(I - T, D, 0) = 0$, and if $K(T, \bar{D}) \cap D \neq \varnothing$ we let ρ be a retraction of X onto $K(T, \bar{D})$ and define

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$$(1) \qquad \deg(I - T, D, 0) = \deg_c(I - T\rho, \rho^{-1}(D), 0),$$

where the right-hand side of (1) means the topological degree defined in [**8**] for multivalued compact vector fields.

Note that the right-hand side of (1) is well defined since $x \in D$ and $x \in T(x)$ if and only if $x \in \mathrm{cl}(\rho^{-1}(D))$ and $x \in T(\rho(x))$, and one may show that this definition is independent of the particular retraction chosen. The combination of retractions and Leray-Schauder degree has been previously used by F. E. Browder in defining a fixed-point index, and by R. D. Nussbaum in defining the degree for single-valued k-set-contractions with $k < 1$. We add that if T is compact (i.e., $T:\bar{D} \to K(X)$ is u.s.c. and $T(\bar{D})$ is relatively compact), then $\deg(I - T, D, 0) = \deg_c(I - T, D, 0)$. Furthermore, this degree has the following properties.

THEOREM 1. *If X, D, and T are as in Definition* 1, *then the degree given by* (1) *is such that*

(a) *if* $\deg(I - T, D, 0) \neq 0$, *then T has a fixed point in D*;

(b) *if $H:\bar{D} \times [0,1] \to K(X)$ is u.s.c., $H(\bar{D} \cap K' \times [0,1])$ is relatively compact where $K' = K(H, \bar{D} \times [0,1])$, and $x \notin H_t(x)$ for $x \in \partial D$ and $t \in [0,1]$, then* $\deg(I - H_0, D, 0) = \deg(I - H_1, D, 0)$;

(c) *if $D = D_1 \cup D_2$, where D_1 and D_2 are open and $D_1 \cap D_2 = \varnothing$, and $x \notin T(x)$ for $x \in \partial D_1 \cup \partial D_2$, then* $\deg(I - T, D, 0) = \deg(I - T, D_1, 0) + \deg(I - T, D_2, 0)$;

(d) *if D is a symmetric neighborhood of the origin and $T:\bar{D} \to K(X)$ is an odd ultimately compact map with $x \notin T(x)$ for $x \in \partial D$, then* $\deg(I - T, D, 0)$ *is odd.*

We add that even in the case of compact multivalued maps the assertion (d) is new for, unlike [**5**], [**8**], we do not require D to be convex.

2. To indicate the usefulness of the topological degree given by Definition 1 we state some examples of ultimately compact maps and some conditions under which the degree is nonzero, a condition which guarantees the existence of fixed points.

If $\{p_\alpha | \alpha \in A\}$ is a family of seminorms which determines the topology on X, $\alpha \in A$ and $\Omega \subset X$, then we define $\gamma_\alpha(\Omega) = \inf\{d > 0 | \Omega$ can be covered by a finite number of sets of p_α-diameter less than $d\}$ and $\chi_\alpha(\Omega) = \{r > 0 | \Omega$ can be covered by a finite number of p_α-balls of radius less than $r\}$. Letting $C = \{f : A \to [0, \infty]\}$, with C ordered pointwise, we define $\gamma : 2^X \to C$ and $\chi : 2^X \to C$ by $\gamma(\Omega)(\alpha) = \gamma_\alpha(\Omega)$ and $\chi(\Omega)(\alpha) = \chi_\alpha(\Omega)$ for each $\alpha \in A$ and $\Omega \subset X$. Then γ and χ are measures of noncompactness which possess the usual properties (see [**14**] for χ) and we let Φ denote either χ or γ. A u.s.c. map $T: D \to CK(X)$ is called Φ-*condensing* if $\Phi(T(A)) \ngeq \Phi(A)$ when $A \subset D$ and $\Phi(A) \neq 0$ and, if $k \in R$, T is called a k-Φ-*con-*

traction if $T(D)$ is bounded and $\Phi(T(A)) \not> k\Phi(A)$ when $A \subset D$. Recall that when X is a Banach space, then $T: D \to CK(X)$ is called *contractive* (*nonexpansive*) if there exist $\alpha \in (0, 1)(\alpha = 1)$ such that

$$d^*(T(x), T(y)) \leqq \alpha d(x, y) \quad \text{for } x, y \in D, \tag{1}$$

where d^* is the Hausdorff metric on $CK(X)$ derived from $d = \| \ \|$. Finally, following [**7**], we say that a u.s.c. map $T: D \to K(X)$ is *generalized condensing* if for each $Q \subset D$ such that $T(Q) \subset Q$ and $Q \backslash \operatorname{clco} T(Q)$ is relatively compact, the set $\bar{Q}$ is compact.

It is clear that every generalized condensing map $T: D \to D$ is ultimately compact if D is closed and convex. Further, if X is also complete (i.e., a Fréchet space), then every $k - \phi$-contraction $T: \bar{D} \to CK(X)$ with $0 < k < 1$ is ϕ-condensing, and every ϕ-condensing map is ultimately compact.

THEOREM 2. *Let $D \subset X$ be convex and open and let $T: \bar{D} \to K(\bar{D})$ be ultimately compact with $K(T, \bar{D}) \neq \varnothing$ and $x \notin T(x)$ for $x \in \partial D$. Then* $\deg(I - T, D, 0) = 1$, *and so T has a fixed point.*

It is not hard to show that if T in Theorem 2 is generalized condensing (and, in particular, if T is ϕ-condensing and X also complete), then $K(T, \bar{D}) \neq \varnothing$ and so Theorem 2 is valid for these classes of maps without the explicit assumption that $K(T, \bar{D}) \neq \varnothing$.

THEOREM 3. *Let X be a Fréchet space, D a neighborhood of* 0, *and $T: \bar{D} \to CK(X)$ a ϕ-condensing map such that*

$$\{\lambda x\} \cap T(x) = \varnothing \quad \text{for } x \in \partial D \text{ and } \lambda \geqq 1. \tag{2}$$

Then T has a fixed point.

Using Theorem 3 one proves the following general fixed point theorem for $1 - \phi$-contractions.

THEOREM 4. *Let X and D be as in Theorem* 3 *and let $T: \bar{D} \to K(X)$ be a $1 - \phi$-contraction. Suppose further that if there is a sequence $\{x_n\} \subset \bar{D}$ with corresponding $y_n \in T(x_n)$ for each n such that $x_n - y_n \to 0$ as $n \to \infty$, then there exists $x \in \bar{D}$ with $x \in T(x)$. If T satisfies* (2), *then T has a fixed point in $\bar{D}$.*

For X a Hausdorff l.c.t.v.s. and T single-valued, Theorem 2 was proved by Sadovsky [**13**]. For X a Banach space and T single-valued and γ-condensing, Theorem 3 was deduced in [**10**] from the index theory for γ-condensing maps developed in [**9**]; for multivalued χ-condensing maps, Theorem 3 includes the result of [**6**], while for multivalued compact maps with $D = B(0, r)$, Theorem 3 was proved in [**4**] for X a Banach space and in [**8**] for D a neighborhood of 0 in a Hausdorff l.c.t.v.s. In case X is Banach and T single-valued, Theorem 4 reduces to Theorem 1 in Petryshyn [**11**].

Our next result extends to condensing maps $T:\bar{D} \to K(X)$ and to symmetric but not necessarily convex sets D the validity of the antipodes theorem established in [**4**], [**8**] for compact multivalued maps and in [**13**] for single-valued condensing maps.

THEOREM 5. *Let X be a Fréchet space, D a symmetric neighborhood of $0 \in X$, and $T:\bar{D} \to CK(X)$ ϕ-condensing. Assume also that*

(3) $\{x - T(x)\} \cap \lambda\{-x - T(-x)\} = \emptyset$ *for* $x \in \partial D$ *and* $\lambda \in [0, 1]$.

Then $\deg(I - T, D, 0)$ *is an odd integer.*

Since a contraction $T:X \to CK(X)$ is α-ball-contractive, $0 < \alpha < 1$, on each bounded set in X, an immediate consequence of Theorems 3 and 5 is the following corollary.

COROLLARY 1. *Let X be a Banach space, D a bounded neighborhood of $0 \in X$, $S:X \to CK(X)$ contractive, and $C:\bar{D} \to CK(X)$ compact. If $T = S + C:\bar{D} \to CK(X)$ satisfies either* (2) *or* (3) *on ∂D, then T has a fixed point in T.*

If in Corollary 1 the map S is defined only on $\bar{D}$, then the conclusion still holds provided that either S is single-valued, or the constant α in (1) is $<\frac{1}{2}$, or X is a Hilbert space and $D = B(0, r)$.

We add in passing that if the Banach space is assumed to have the so-called Opial property, D is weakly compact, and $C:\bar{D} \to CK(X)$ is completely continuous, then Corollary 1 also holds for $S:X \to CK(X)$ nonexpansive.

We end our note with the following mapping theorem which extends the corresponding result of Ma [**8**] for multivalued compact maps.

THEOREM 6. *Let X be a Fréchet space, $D \subset X$ an open set, and $T:D \to K(X)$ $k - \phi$-contractive with $k < 1$. If T is a boundary map in the sense of Ma* [**8**], *then* $(I - T)(D)$ is open.

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