NONLOCALLY FLAT EMBEDDINGS, SMOOTHINGS, AND GROUP ACTIONS

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Introduction. This note announces some methods and results concerned with piecewise linear (PL) embeddings of manifolds, in codimension two, that are not necessarily locally flat, i.e., not locally smoothable. Theorems 4 and 1 are PL embedding theorems in codimension two. They are analogous to PL embedding theorems of [H] (see also [W], especially for nonsimply connected cases) for higher codimensions; recall that for codimension greater than two, a PL embedding of a manifold is always locally flat. The present results have a simpler form in odd dimensions.

Our methods can also be applied to the study of equivariant submanifolds, as in Theorems 2 and 3 below. In the papers [CS3], [CS4], it was determined which free actions of a cyclic group Z_p on a sphere S^{2n-1} are the restrictions of a given free Z_p action T on S^{2n+1} to an invariant sphere for which the action carries the fibers of a closed tubular neighborhood linearly to fibers. Using a new embedding theorem, it is determined here which free cyclic actions on S^{2n-1}, $n \geq 2$, are restrictions of T to an invariant sphere, without any condition on the behavior of the action in the neighborhood of the invariant sphere.

The proof of Theorem 3 leads to examples of the following type: Let $n \geq 4$ be an integer not of the form $2^r - 1$ or $2^r - 3$, and let P^n denote real projective n-space. Then there is a PL manifold Q^{n+2}, homotopy equivalent to P^{n+2}, with the following properties:

(i) There are PL embeddings of P^n in Q^{n+2}.

(ii) If $f: P^n \to Q^{n+2}$ is a PL embedding, let $X_f \subset P^n$ be the set of points at which f is not locally flat. Then the natural map $H_i(X_f; Z_2) \to H_i(P^n; Z_2)$ is onto for $0 \leq i \leq n - 2$.

Singularities of PL embeddings will be studied further in [CS5].

We also compute the PL, but not necessarily locally flat, s-cobordism classes of PL embeddings, homotopic to the zero-section, of a manifold in a disk bundle. For example, for the n-torus $T^n = S^1 \times \cdots \times S^1$, $n \geq 3$, the set of such classes is in one-to-one correspondence with a finite nontrivial direct sum of subgroups of knot cobordism groups consisting of knots with vanishing index or Arf invariant. In contrast, it was

shown in [CS3], [CS4] that for n odd, for example, the group $C_{PL}(T^n)$ of PL locally flat cobordism classes of embeddings of T^n in $T^n \times D^2$, homotopic to the zero-section, is a finite sum of copies of the *quotients* of knot cobordism groups by the subgroups of knots of vanishing index or Arf invariant. Thus the group $C_{PL}(T^n)$ is a finite sum of copies of Z and Z_2.

For a PL manifold W^{n+2} of the homotopy type of a PL n-manifold, $n \geq 3$, we show that if n is odd, or if $\pi_1 W$ vanishes or is infinite cyclic[2] and W is orientable, then every closed manifold M (assumed orientable if $\pi_1 W^{2k} = Z$) homotopy equivalent to W occurs as a spine of W; i.e., M can be PL embedded in W by a simple homotopy equivalence, so that W will be a regular neighborhood of the image. One consequence is the result of Kato-Matsumoto [KM] on the existence, in the simply-connected case, of a spine that is locally flat, for n odd, and locally flat except at one point, for n even. (This result also follows easily from the codimension two splitting theorems of [CS4, §8].) On the other hand, given W as above one can find a homotopy equivalence $h: M \to W$ with the following properties: (i) h is homotopic to a PL embedding, and (ii) for every embedding $f: W \to M$ homotopic to h, the set X_f of nonlocally flat points is large in the sense that $X_f \cap F(V_\alpha)$ is nonempty for every component V_α of a characteristic variety of Sullivan [S2], $F: V \to M$ of M.

Statement of some results. Below all manifolds are PL, connected, and compact, and all embeddings are PL, i.e., simplicial after suitable subdivision, but not necessarily locally flat. Related topological results will be discussed elsewhere. More results and proofs will appear in [CS5].

THEOREM 1. *Let $f: N^n \to W^{n+2}$ be a PL embedding and let $g: M \to N$ be a homotopy equivalence of closed PL manifolds. Assume that n is odd, or that $\pi_1 N$ vanishes, or that $\pi_1 N$ is infinite cyclic and N and W are orientable.[3,4] Then fg is homotopic to a PL embedding.*

(It suffices for Theorem 1 that N be a PL embedded Poincaré complex, $n \neq 2$.)

THEOREM 2. *Let (Σ^{2n-1}, T_1), (Σ^{2n+1}, T_2), $n \neq 2$, be spheres, equipped with free PL actions of S^1, T_1, and T_2. Then there is an equivariant PL embedding of (Σ^{2n-1}, T_1) in (Σ^{2n+1}, T_2).*

However, an embedding that induces a linear action of S^1 on the normal bundle exists, for $n \geq 3$, if and only if (Σ^{2n+1}, T_2) is a suspension of (Σ^{2n-1}, T_1), which will not in general be the case [S2].

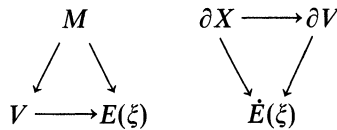[2]This uses the homology splitting principle of [CS4, V] for Γ-groups.
[3]The result remains valid if $\pi_1(R - N)$ is abelian, R a regular neighborhood of N in W.
[4]That is, the quotient space is a PL manifold.

THEOREM 3. *Let* (Σ_1^{2n-1}, T_1) *and* (Σ_2^{2n+1}, T_2) *be spheres equipped with free PL actions of* Z_p, T_1 *and* T_2. *Then there is an equivariant embedding of* (Σ_1, T_1) *in* (Σ_2, T_2) *if and only if* $\exists$ *a suspension* (Σ_0, T_0) *of* T_1 (*see* [W]) *with* Σ_0/T_0 *homotopy equivalent to* Σ^{2n+1}/T_2. *However, for* p *odd, this embedding could be constructed so that* T_2 *restricts to a linear* Z_p *action on the normal bundle of* Σ_1^{2n-1} *in* Σ_2^{2n+1} *if and only if* Σ_1/T_1 *is normally cobordant to a desuspension of* Σ_2/T_2, $n \geq 3$ [CS3], [CS4].

The more complicated situation for p even is studied in [CS3], [CS4]. (See also [CS6].)

To illustrate the main ideas involved in proving theorems such as the above, we first introduce the notion of a (homology) Poincaré embedding in codimension two. By a Poincaré embedding of the closed manifold M^n in the closed manifold W^{n+2} we mean a (simple) homotopy equivalence $f: W \to (E(\xi) \cup_{\dot{E}(\xi)} Y)$, where $E(\xi)$ is the total space of the associated disk fibration of a spherical fiber space ξ over M, and where $(Y, E(\xi))$ is a Poincaré pair (even only with coefficients in $Z\pi_1 W$). We say f is realized by a PL embedding if there are PL manifolds $(V, \partial V)$, $(X, \partial X)$, $\partial X = \partial V$, equipped with a PL embedding $M \to V$; a homology equivalence, inducing an isomorphism on π_1, with coefficients in the local system of $\pi_1 M$, $(V, \partial V) \to (E(\xi), \dot{E}(\xi))$; and a homology equivalence with coefficients in the local system of $\pi_1(W)$, $(X, \partial X) \to (Y, \dot{E}(\xi))$; so that the diagrams

$$
\begin{array}{ccc}
& M & \qquad \partial X \longrightarrow \partial V \\
\swarrow \;\; \searrow & & \searrow \;\; \swarrow \\
V \longrightarrow E(\xi) & & \dot{E}(\xi)
\end{array}
$$

are homotopy commutative and so that the induced map $(V \cup_{\partial V} X) \to W$ is homotopic to a PL homeomorphism. These notions can be relativized if W and/or M are manifolds with boundary; on ∂W, f is required to be a (simple) homology equivalence over $Z\pi_1 W$. A PL embedding of a manifold homotopy equivalent to M^n, in W^{n+2}, determines a Poincaré embedding of M in W.

Given a Poincaré embedding M in W, as above, with $\pi_1 M \to \pi_1 W$ surjective, one can find a "concordant" Poincaré embedding with $\pi_1 Y$ a cyclic extension of $\pi_1 W$ [CS5]. More generally, this can be done if and only if $Z \otimes_{Z[\pi_1 M]} Z[\pi_1 W]/\{[M] \cdot \pi_2(W)\}$, which is isomorphic to the commutator quotient of the kernel of $\pi_1 Y \to \pi_1 W$, is a cyclic group.

THEOREM 4. *Suppose we are given a Poincaré embedding of* M^n *in* W^{n+2},

$n \geq 3$. If n is odd or if, for Y as above, $\pi_1 Y$ is isomorphic to $\pi_1 W$ via the natural map, or if $\pi_1 W$ vanishes or is infinite cyclic[5] and $\pi_1 Y$ is a cyclic extension of $\pi_1 W$, then this Poincaré embedding can be realized by a PL embedding.

Note. If $\pi_1 M = 0$ and n is odd, then for only one class of manifolds (N, g) in the set ht(M), with $g : N \to M$ a homotopy equivalence, will the composition of the given Poincaré embedding with g be realizable by a locally flat embedding. If n is even and $\pi_1 M = 0$, this composite will be realizable as a PL embedding which is locally flat except at one point for only one class of manifolds (N, g) in the homotopy triangulations ht(M). For $\pi_1 M \neq 0$, there are examples involving a torus for which no such embedding exists. For n even, the general obstruction to realizing a Poincaré embedding by a PL embedding can be described in the terms of the Γ-groups introduced in [CS2], [CS3], [CS4].

Outline of the proof of Theorem 4. Using induced PL neighborhoods [S1], a classifying space BRN_2 $(BSRN_2)$ for concordance classes of (oriented) PL codimension two thickenings (regular neighborhoods) of (oriented) manifolds can be constructed. There are natural maps $BSO_2 \to BSRN_2 \to BSG_2 = BSO_2$. Let G_2/RN_2 be the fiber of $BSRN_2 \to BSO_2$. There is a natural map $G_2/RN_2 \to G/PL$.

An analogous PL embedding theorem for higher codimensions is proved ([H], [W, 11.3]) using the isomorphism $G_k/PL_k = G/PL$, $k \geq 3$, to lift (stable) normal invariants to G_k/PL_k and an argument showing that a resulting surgery problem can be solved. A related construction using (i) and (iii) of Theorem 5 reduces the argument to showing, using methods of [CS4], that an induced homology surgery problem can be solved. Results of [CS4] applied there to similar problems for locally flat embeddings show that the surgery problem can always be solved when n is odd. For n even, by a further computation [CS4, V] of Γ-groups, the homology surgery obstruction can be killed by varying on the top cell of M the lift of a normal invariant into G_2/RN_2. Note that a trivial normal invariant can be lifted to a trivial map, for n odd, and to a map trivial in the complement of a point, for n even, for which the homology surgery problem can be solved.

Let $\Gamma^{\varepsilon(k)}$, $\varepsilon(k) = \text{sign}(-1)^k$, denote the groups $\Gamma_{2k+2}(\phi_0)$ of [CS3], isomorphic to high dimensional knot cobordism groups. Let γ be the Fox-Milnor cobordism group of knots $S^1 \subset S^3$. Let Y be the fiber of $K(\gamma; 2) \times K(\Gamma_{(2)}^+; 4) \to K(Z_2; 4)$ given by the cohomology class $\alpha \otimes 1 + 1 \otimes \beta$, α the square of the class given by the Arf invariant and β given by the mod two reduction of $\frac{1}{8}$ (index). For a group G, let $G_{(2)}$ and $G_{(\text{odd})}$

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[5]The result remains valid if $\pi_1 W$ is free abelian and $\pi_1 Y$ is abelian.

denote localizations at and away from 2, and $G_Q = G \otimes Q$.

THEOREM 6. (i) $BSRN_2$ is homotopy equivalent to $BSO_2 \times G_2/RN_2$.
(ii) There is a fiber product diagram

$$G_2/RN_2 \longrightarrow Y \times \prod_{i>2} K(\Gamma_{(2)}^{\varepsilon(i)}; 2i)$$

$$\downarrow \qquad\qquad\qquad\qquad\qquad\qquad \downarrow \lambda$$

$$BO \otimes \Gamma_{(odd)}^{+} \times \Omega^2(BO \otimes \Gamma_{(odd)}^{-}) \xrightarrow{\rho} K(\Gamma_{(odd)}^{-}; 2) \times \prod_{i>1} K(\Gamma_Q^{\varepsilon(i)}; 2i),$$

where ρ is essentially the Pontrjagin character and λ is induced by natural maps $\gamma \to \Gamma^-$ and $\Gamma_{(2)}^{\pm} \to \Gamma_Q^{\pm}$.
(iii) The natural map $BSRN_2 \to G/PL$ splits (unnaturally).

Results similar to (i) and (ii) were announced by L. Jones [J], "modulo low dimensional complications". The proof is similar to Sullivan's "characteristic variety theorem," which may be interpreted as the description of G/PL associated to the Browder-Novikov surgery theory.

In the present case, surgery theory is replaced by the homology surgery theory of [CS4]. A general "characteristic variety theorem" for suitable periodic spaces X, with period 4 and with retractions of bordism groups to homotopy satisfying suitable product formulas, seems to have been known for some time, at least to Morgan and to Jones.

The Eilenberg obstruction to contracting a map to G_2/RN_2 are identified with the Noguchi obstructions [N] to making an embedding locally flat.

Due to the absence of a simple description of the H-structure of $BSRN_2$, (i) requires careful geometry.

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