BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 79, Number 3, May 1973

PERIODIC AND HOMOGENEOUS STATES ON A VON NEUMANN ALGEBRA. III

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Communicated by Jacob E. Feldman, September 12, 1972

In this paper, we will show with a fairly complete proof that most of the results in [**10**] hold for homogeneous periodic states on a factor without the assumption of *inner* homogeneity. As an application, we will see that nonisomorphic ergodic automorphisms $\tilde{\theta}$ of $\mathscr{Z}_0$ give rise to nonisomorphic factors $\mathscr{R}(\mathscr{M}_0, \theta)$ of type III. We keep most of the terminology and the notations in [**9**] and [**10**].

We consider an arbitrary pair of homogeneous periodic states φ and ψ on a factor $\mathscr{M}$ of the same period, say $T > 0$. Let $\kappa = e^{-2\pi/T}$, $0 < \kappa < 1$. We denote by $\mathscr{M}_n^{\varphi,\psi}$ the set of all $x \in \mathscr{M}$ such that $\rho_t(x) = \kappa^{int}x$, $t \in \boldsymbol{R}$, which was denoted by $\mathscr{V}_n$ in [**10**]. With this alternation of the notation, we first note that Lemmas 1 through 6 remain valid without the assumption of inner homogeneity. Since $\mathscr{M}_0^{\varphi}$ and $\mathscr{M}_0^{\psi}$ are no longer factors, we have to analyze more carefully the relation between $\mathscr{M}_0^{\varphi}, \mathscr{M}_0^{\varphi,\psi}$ and $\mathscr{M}_0^{\psi}$. We denote by $\mathscr{Z}_0^{\varphi}$ and $\mathscr{Z}_0^{\psi}$ the center of $\mathscr{M}_0^{\varphi}$ and $\mathscr{M}_0^{\psi}$ respectively, and by u_φ and u_ψ the isometries in $\mathscr{M}_1^{\varphi}$ and $\mathscr{M}_1^{\psi}$ respectively which give rise to isomorphisms θ_φ and θ_ψ of $\mathscr{M}_0^{\varphi}$ and $\mathscr{M}_0^{\psi}$ onto $e_\varphi \mathscr{M}_0^{\varphi} e_\varphi$ and $e_\psi \mathscr{M}_0^{\psi} e_\psi$ respectively, where $e_\varphi = u_\varphi u_\varphi^*$ and $e_\psi = u_\psi u_\psi^*$. We also denote by $\tilde{\theta}_\varphi$ and $\tilde{\theta}_\psi$ the automorphisms of $\mathscr{Z}_0^{\varphi}$ and $\mathscr{Z}_0^{\psi}$ induced by θ_φ and θ_ψ respectively. Since $\mathscr{M}$ is a factor, we know from [**9**, Proposition 9] that $\tilde{\theta}_\varphi$ and $\tilde{\theta}_\psi$ are both ergodic.

LEMMA 1. *For each* $n \in \boldsymbol{Z}$, *we have*

$$(1) \qquad \mathscr{M}_{n-1}^{\varphi,\psi} = u_\varphi^* \mathscr{M}_n^{\varphi,\psi} \quad \textit{and} \quad \mathscr{M}_{n+1}^{\varphi,\psi} = \mathscr{M}_n^{\varphi,\psi} u_\psi.$$

PROOF. From [**10**, Lemma 5], it follows that $\mathscr{M}_{n-1}^{\varphi,\psi} \supset u_\varphi^* \mathscr{M}_n^{\varphi,\psi}$; so

$$\mathscr{M}_{n-1}^{\varphi,\psi} = u_\varphi^* u_\varphi \mathscr{M}_{n-1}^{\varphi,\psi} \subset u_\varphi^* \mathscr{M}_n^{\varphi,\psi} \subset \mathscr{M}_{n-1}^{\varphi,\psi}.$$

Hence we get $\mathscr{M}_{n-1}^{\varphi,\psi} = u_\varphi^* \mathscr{M}_n^{\varphi,\psi}$. By symmetry, the assertion for u_ψ follows. Q.E.D.

LEMMA 2. *For any nonzero projections* $p \in \mathscr{M}_0^{\varphi}$ *and* $q \in \mathscr{M}_0^{\psi}$, *we have*

$$p\mathscr{M}_n^{\varphi,\psi} \neq \{0\} \quad \textit{and} \quad \mathscr{M}_n^{\varphi,\psi} q \neq \{0\}.$$

AMS (MOS) subject classifications (1970). Primary 46L10.
Key words and phrases. von Neumann algebra, modular automorphism group, periodic state, homogeneous state.
[1]This work is supported in part by NSF Grant No. GP-33696X.

PROOF. Let $\mathscr{I}_n$ be the set of all $x \in \mathscr{M}_0^\varphi$ with $x\mathscr{M}_n^{\varphi,\psi} = \{0\}$. By [**10**, Lemma 5], $\mathscr{I}_n$ is a σ-weakly closed ideal of $\mathscr{M}_0^\varphi$, so that there exists a projection $z_n \in \mathscr{Z}_0^\varphi$ such that $\mathscr{I}_n = \mathscr{M}_0^\varphi z_n$. If $x \in \mathscr{I}_n$, then $x\mathscr{M}_{n+1}^{\varphi,\psi} = x\mathscr{M}_n^{\varphi,\psi}u_\psi = \{0\}$ by (1), which means that $\mathscr{I}_n \subset \mathscr{I}_{n+1}$; so $z_n \leqq z_{n+1}$. We have

$$\tilde{\theta}_\varphi(z_n)e_\varphi\mathscr{M}_{n+1}^{\varphi,\psi} = u_\varphi z_n u_\varphi^* \mathscr{M}_{n+1}^{\varphi,\psi} = u_\varphi z_n \mathscr{M}_n^{\varphi,\psi} = \{0\},$$

so that $\tilde{\theta}_\varphi(z_n)e_\varphi \leqq z_{n+1}$; hence $\tilde{\theta}_\varphi(z_n) \leqq z_{n+1}$. Conversely, we have

$$\tilde{\theta}_\varphi^{-1}(z_{n+1})\mathscr{M}_n^{\varphi,\psi} = u_\varphi^* z_{n+1} u_\varphi \mathscr{M}_n^{\varphi,\psi} \subset u_\varphi^* z_{n+1}\mathscr{M}_{n+1}^{\varphi,\psi} \subset \{0\}.$$

Hence we get $\tilde{\theta}_\varphi^{-1}(z_{n+1}) \leqq z_n$, so that $z_{n+1} \leqq \tilde{\theta}_\varphi(z_n)$. Thus we have $z_{n+1} = \tilde{\theta}_\varphi(z_n)$. Hence $z_n \leqq \tilde{\theta}_\varphi(z_n)$. The equality $\varphi(z_n) = \varphi \circ \tilde{\theta}_\varphi(z_n)$ implies that z_n must be either 0 or 1. Since $\mathscr{M}_n^{\varphi,\psi} \neq \{0\}$, we have $z_n = 0$. Hence $p\mathscr{M}_n^{\varphi,\psi} \neq \{0\}$. By symmetry, $\mathscr{M}_n^{\varphi,\psi}q \neq \{0\}$. Q.E.D.

LEMMA 3. *Let v_1 and v_2 be partial isometries in $\mathscr{M}_n^{\varphi,\psi}$ with initial projections q_1, q_2 and final projections p_1, p_2 respectively. Then the following statements are equivalent*:

(i) *p_1 and p_2 are centrally orthogonal in $\mathscr{M}_0^\varphi$, i.e. $p_1\mathscr{M}_0^\varphi p_2 = \{0\}$*;
(ii) *q_1 and q_2 are centrally orthogonal in $\mathscr{M}_0^\psi$, i.e. $q_1\mathscr{M}_0^\psi q_2 = \{0\}$.*

PROOF. By symmetry, we have only to prove (i) $\Rightarrow$ (ii). Suppose $q_1\mathscr{M}_0^\psi q_2 \neq \{0\}$. Let x be an element in $\mathscr{M}_0^\psi$ with $q_1 x q_2 \neq \{0\}$. We have then $v_1^* v_1 x v_2^* v_2 \neq 0$, so that $v_1 x v_2^* \neq 0$. Hence $p_1 v_1 x v_2^* p_2 = v_1 x v_2^* \neq 0$. But this is impossible because $v_1 x v_2^*$ is in $\mathscr{M}_0^\varphi$ by [**10**, Lemma 5]. Q.E.D.

Suppose $\{v_i\}_{i\in I}$ is a maximal family of partial isometries in $\mathscr{M}_n^{\varphi,\psi}$ such that the initial projections $q_i = v_i^* v_i$ are centrally orthogonal in $\mathscr{M}_0^\psi$. Let $p_i = v_i v_i^*$. By Lemma 3, $\{p_i\}_{i\in I}$ are centrally orthogonal in $\mathscr{M}_0^\varphi$. Hence $v = \sum_{i\in I} v_i$ is a partial isometry in $\mathscr{M}_n^{\varphi,\psi}$. Let $p = vv^*$ and $q = v^*v$. By Lemma 3, we conclude that the central supports of p and q in $\mathscr{M}_0^\varphi$ and $\mathscr{M}_0^\psi$ are both the identity. Therefore, there exists an isomorphism σ_v of $\mathscr{Z}_0^\psi$ onto $\mathscr{Z}_0^\varphi$ such that

$$\sigma_v(a)p = vav^*, \qquad a \in \mathscr{Z}_0^\psi. \tag{2}$$

LEMMA 4. *For every projection $f \in \mathscr{Z}_0^\psi$, $\sigma_v(f)$ is characterized as the smallest projection $e \in \mathscr{Z}_0^\varphi$ such that $e\mathscr{M}_n^{\varphi,\psi}f = \mathscr{M}_n^{\varphi,\psi}f$.*

PROOF. Let e be the smallest projection in $\mathscr{Z}_0^\varphi$ with $e\mathscr{M}_n^\varphi f = \mathscr{M}_n^{\varphi,\psi}f$. We have then $evf = vf$, so that

$$\sigma_v(f)p = vfv^* = evfv^*e = e\sigma_v(f)pe = \sigma_v(f)ep.$$

Hence $(\sigma_v(f) - \sigma_v(f)e)p = 0$. Since the central support of p is 1, we have $\sigma_v(f) = \sigma_v(f)e$; that is, $\sigma_v(f) \leqq e$. If $e - \sigma_v(f) \neq 0$, then there exists an $x \in \mathscr{M}_n^{\varphi,\psi}$ with $[e - \sigma_v(f)]xf = x \neq 0$. Let $x = wh = kw$ be the left and right polar decomposition of x. As in the arguments (8) in [**19**], $w \in \mathscr{M}_n^{\varphi,\psi}$.

By the choice of x, we have $ww^* \leqq e - \sigma_v(f)$ and $w^*w \leqq f$. On the other hand, we have $vf = \sigma_v(f)vf$, so that $(vf)(vf)^* \leqq \sigma_v(f)$ and $(vf)^*(vf) = fv^*vf = fq$. Hence the central support of $(vf)^*(vf)$ in $\mathscr{M}_0^\psi$ is f. But this is impossible by Lemma 3 because the central supports of $(vf)(vf)^*$ and ww^* in $\mathscr{M}_0^\varphi$ are orthogonal. Thus we get $\sigma_v(f) = e$. Q.E.D.

Therefore, the isomorphism σ_v does not depend on the choice of v, but only on $n \in \mathbf{Z}$; so we denote it by σ_n.

LEMMA 5. *For each* $n \in \mathbf{Z}$, *we have*

$$\sigma_n \circ \tilde{\theta}_\psi = \sigma_{n+1} = \tilde{\theta}_\varphi \circ \sigma_n. \tag{3}$$

PROOF. Let f be an arbitrarily fixed projection in $\mathscr{Z}_0^\psi$. Let $e_n = \sigma_n(f) \in \mathscr{Z}_0^\varphi$. We have then

$$e_{n+1}u_\varphi \mathscr{M}_n^{\varphi,\psi}f = u_\varphi \mathscr{M}_n^{\varphi,\psi}f,$$

$$u_\varphi^* e_{n+1}u_\varphi \mathscr{M}_n^{\varphi,\psi}f = \mathscr{M}_n^{\varphi,\psi}f.$$

Hence we have $\tilde{\theta}_\varphi^{-1}(e_{n+1}) \geqq e_n$; equivalently, $e_{n+1} \geqq \tilde{\theta}_\varphi(e_n)$.

On the other hand, putting $z = 1 - \tilde{\theta}_\varphi(e_n)$, we have

$$u_\varphi^* z u_\varphi \mathscr{M}_n^{\varphi,\psi}f = (1 - e_n)\mathscr{M}_n^{\varphi,\psi}f = \{0\};$$

$$ze_\varphi \mathscr{M}_{n+1}^{\varphi,\psi}f = zu_\varphi \mathscr{M}_n^{\varphi,\psi}f = u_\varphi u_\varphi^* z u_\varphi \mathscr{M}_n^{\varphi,\psi}f = \{0\}.$$

Hence we have $ze_\varphi \leqq (1 - e_{n+1})$; so $z \leqq 1 - e_{n+1}$. Therefore we get $\tilde{\theta}_\varphi(e_n) \geqq e_{n+1}$. Thus we have $e_{n+1} = \tilde{\theta}_\varphi(e_n)$; that is, $\tilde{\theta}_\varphi \circ \sigma_n(f) = \sigma_{n+1}(f)$ for every projection $f \in \mathscr{Z}_0^\psi$, which means that $\sigma_{n+1} = \tilde{\theta}_\varphi \circ \sigma_n$.

By symmetry, the other half of our assertion follows. Q.E.D.

COROLLARY 6. *The ergodic automorphisms* $\tilde{\theta}_\varphi$ *of* $\mathscr{Z}_0^\varphi$ *and* $\tilde{\theta}_\psi$ *of* $\mathscr{Z}_0^\psi$ *are isomorphic.*

LEMMA 7. *If* v *is a partial isometry in* $\mathscr{M}_n^{\varphi,\psi}$ *such that the initial projection* $q = v^*v$ *and the final projection* $p = vv^*$ *have the central support* 1, *then we have*

$$p^\natural = \alpha\kappa^n\sigma_n(q^\natural), \tag{4}$$

where α *is the real number defined in* [**10**].

PROOF. Consider a faithful state $\varphi \circ \sigma_n$ on $\mathscr{Z}_0^\psi$. Then $\varphi \circ \sigma_n \circ \tilde{\theta}_\psi = \varphi \circ \tilde{\theta}_\varphi \circ \sigma_n = \varphi \circ \sigma_n$, so that $\varphi \circ \sigma_n$ is $\tilde{\theta}_\psi$-invariant; hence $\varphi \circ \sigma_n$ is a scalar multiple of ψ on $\mathscr{Z}_0^\psi$ by the ergodicity of $\tilde{\theta}_\psi$. But $\varphi \circ \sigma_n$ and ψ are both states, so that $\varphi \circ \sigma_n = \psi$ on $\mathscr{Z}_0^\psi$.

We have next, for every $a \in \mathscr{Z}_0^\psi$,

$$\begin{aligned}\psi(a\sigma_n^{-1}(p^\natural)) &= \varphi(\sigma_n(a\sigma_n^{-1}(p^\natural))) = \varphi(\sigma_n(a)p^\natural)\\ &= \varphi(\sigma_n(a)p) = \varphi(vav^*)\\ &= \alpha\kappa^n\psi(av^*v) \qquad \text{by [10, Lemma 4]}\\ &= \alpha\kappa^n\psi(aq) = \alpha\kappa^n\psi(aq^\natural).\end{aligned}$$

Thus, we get $\sigma_n^{-1}(p^\natural) = \alpha\kappa^n q^\natural = \alpha\kappa^n q^\natural$, equivalently $p^\natural = \alpha\kappa^n\sigma_n(q^\natural)$.

Making use of the similar arguments as in Lemma 2, we conclude the following:

LEMMA 8. *If $p \in \mathcal{M}_0^\varphi$ and $q \in \mathcal{M}_0^\varphi$ are projections with central support e and f in $\mathcal{M}_0^\varphi$ and $\mathcal{M}_0^\psi$ respectively, then $p\mathcal{M}_n^{\varphi,\psi}q = \{0\}$ if and only if $e\sigma_n(f) = 0$.*

Now, let $\{v_i\}_{i\in I}$ be a maximal family of partial isometries in $\mathcal{M}_n^{\varphi,\psi}$ such that the initial projections $q_i = v_i^*v_i$ and the final projections $p_i = v_iv_i^*$ are orthogonal respectively. Let $v = \sum_{i\in I} v_i$, $p = \sum_{i\in I} p_i$ and $q = \sum_{i\in I} q_i$. By Lemma 3, the central supports of p in $\mathcal{M}_0^\varphi$ and q in $\mathcal{M}_0^\psi$ are respectively the identity. By maximality, we have $(1-p)\mathcal{M}_n^{\varphi,\psi}(1-q) = \{0\}$. Let e and f be the central supports of p in $\mathcal{M}_0^\varphi$ and q in $\mathcal{M}_0^\psi$ respectively. By Lemma 8, $e\sigma_n(f) = 0$. On the other hand, we have $p^\natural = \alpha\kappa^n\sigma_n(q^\natural)$ by Lemma 7. Hence $p^\natural \leqq \alpha\kappa^n \leqq 1$ if $n \geqq 1$. Hence we have $e = 1$, so that $f = 0$; so $q = 1$. Hence v must be an isometry if $n \geqq 1$. Similarly, if $n \leqq 0$, then v is a co-isometry. For $n = 0$, v is unitary if and only if $\alpha = 1$. Thus we reach the following conclusion:

THEOREM 9. *If φ and ψ are homogeneous periodic states on a factor $\mathcal{M}$ with same period, then there exists isometries u and v in $\mathcal{M}$ such that*

$$\begin{aligned}\psi(x) &= \varphi(uxu^*)/\varphi(uu^*),\\ \varphi(x) &= \psi(vxv^*)/\psi(vv^*), \qquad x \in \mathcal{M};\\ p = uu^* &\in \mathcal{M}_0^\varphi \quad \text{and} \quad q = vv^* \in \mathcal{M}_0^\psi.\end{aligned}$$

From this theorem, we can conclude that Theorems 8 through 10 in [10] hold for homogeneous periodic states φ, ψ with the same period and/or for projections p and q with uniform relative dimensions.

Let $\mathcal{F}$ be a hyperfinite II_1-factor and $\mathcal{A} = L^\infty(0, 1)$. Let $\mathcal{M}_0 = \mathcal{F} \otimes \mathcal{A}$. For $0 < \kappa < 1$, we choose a projection $f \in \mathcal{F}$ with $\tau(f) = \kappa$, where τ is the canonical trace of $\mathcal{F}$. Let θ be a fixed isomorphism of $\mathcal{F}$ onto $f\mathcal{F}f$. For each $\sigma \in \mathrm{Aut}(\mathcal{F})$, let $\theta_\sigma = \theta \circ \sigma$. Let $\tilde{\theta}$ be an ergodic automorphism of $\mathcal{A}$ with invariant faithful normal state μ. Changing $\tilde{\theta}$ under an automorphism of $\mathcal{A}$, we may assume that μ is given by the Lebesgue measure on $(0, 1)$. Let $\varphi_0 = \tau \otimes \mu$. We obtain then a factor $\mathcal{R}(\mathcal{M}_0, \theta_\sigma \otimes \tilde{\theta}, \varphi_0)$ as described in [9]. We denote it by $\mathcal{M}(\kappa, \sigma, \tilde{\theta})$.

THEOREM 10. *We choose* $\sigma_1, \sigma_2 \in \mathrm{Aut}(\mathscr{F})$ *and ergodic automorphisms* $\tilde{\theta}_1$ *and* $\tilde{\theta}_2$ *of* $\mathscr{A}$ *and fix* κ. *A necessary and sufficient condition for* $\mathscr{M}(\kappa, \sigma_1, \tilde{\theta}_1) \cong \mathscr{M}(\kappa, \sigma_2, \tilde{\theta}_2)$ *is that*

(i) $\tilde{\theta}_1$ *and* $\tilde{\theta}_2$ *are isomorphic as ergodic automorphisms of* $\mathscr{A}$;

(ii) *there exist a projection* p *in* $\mathscr{F}$ *with* $p \geqq f$, *a partial isometry* w *in* $\mathscr{F}$ *and an isomorphism* ρ *of* $\mathscr{F}$ *onto* $p\mathscr{F}p$ *such that*

$$w\theta \circ \sigma_1 \circ \rho(x)w^* = \rho \circ \theta \circ \sigma_2(x);$$

$$\theta \circ \sigma_1 \circ \rho(x) = w^*\rho \circ \theta \circ \sigma_2(x)w, \qquad x \in \mathscr{M}.$$

Furthermore, if $\tilde{\theta}_1$ *and* $\tilde{\theta}_2$ *have no point spectrum other than* 1, *then the isomorphy* $\mathscr{M}(\kappa_1, \sigma_1, \tilde{\theta}_1) \cong \mathscr{M}(\kappa_2, \sigma_2, \tilde{\theta}_2)$ *implies that* $\kappa_1 = \kappa_2$ *as well as* (i) *and* (ii).

ACKNOWLEDGEMENT. The author would like to thank Professor H. Araki for enjoyable discussions on the subject of this paper; Dr. A. Connes for his kindness informing the author of his new results on which this work depends; Professor H. A. Dye for his constant encouragement; and the members of the functional analysis seminar at Queen's University where the related topics were extensively discussed.

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