BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 79, Number 3, May 1973

SOME L GROUPS OF FINITE GROUPS

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Communicated by Hyman Bass, October 24, 1972

If π is a finite group, define the modified Whitehead group $\mathrm{WH}'(\pi)$ to be the quotient of $\mathrm{Im}(K_1(\boldsymbol{Z}\pi) \to K_1(\boldsymbol{Q}\pi))$ (the group of reduced norms of invertible matrices over $\boldsymbol{Z}\pi$) by the classes of $\pm g$, $g \in \pi$. Using classes in this, we have a concept of 'near-simple' homotopy equivalence, and a family of surgery obstruction groups, which we denote in this paper by $L_n(\pi)$.

Roughly speaking, $L_0(\pi)$ (resp. $L_2(\pi)$) is the Grothendieck group of nonsingular hermitian (resp. skew hermitian) forms over the group ring $\boldsymbol{Z}\pi$, with involution defined by $g \mapsto w(g)g^{-1}$ $(g \in \pi)$ for some homomorphism $w:\pi \to \{\pm 1\}$; $L_1(\pi)$ (resp. $L_3(\pi)$) is the commutator quotient group of the (stable) unitary group of such forms. The precise definition is given in [**9**] or (better) [**10**]. The 'orientable' case π^+ is when w is trivial.

The object of this note is to announce the following calculations. For any abelian group G, we write ${}_2G$ and G_2 for the kernel and cokernel of $2:G \to G$.

(i) π *of odd order.* Write $R(\pi)$ for the complex representation ring of π, $\bar{x}$ for the complex conjugate of x.

$L_{2k+1}(\pi) = 0$. *The signature map on* $L_{2k}(\pi)$ *has kernel* 0 (k *even*), $\boldsymbol{Z}_2$ (k *odd*), *and image* $\{4(x + (-1)^k\bar{x}) : x \in R(\pi)\}$.

(ii) π *abelian.* Write N for the order of π, r for the 2-rank, s for the number of direct summands of order 2.

Special case. For some $x \in \pi, x^2 = 1$ *and* $w(x) = -1$. $L_n(\pi) \cong L_n(\boldsymbol{Z}_2^-) \oplus E$, *where E is an elementary 2-group of rank* $(N/2 - N/2^r - r + 1)$. $L_n(\boldsymbol{Z}_2^-) = 0$ (n *odd*) $= \boldsymbol{Z}_2$ (n *even*).

General case. There is no such x. The image of the signature map on $L_n(\pi)$ *is as in* (i) *for n even, π orientable, and* 0 *otherwise. The kernel has exponent* 2 *and rank*

$$
\begin{array}{ll}
2^r - 1 - r - \binom{s}{2} & n \equiv 0, 1(4), \\
1, & n \equiv 2(4), \\
2^r - 1, & n \equiv 3(4)\ \textit{orientable},
\end{array}
$$

exponent 2 *or* 4 *and order* $2^{(2^r + 2^{r-1} - 1)}$ *in the other case.*

(iii) π *dihedral of order* $2p$ (p *an odd prime*). Let K_p denote the maximal

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AMS (*MOS*) *subject classifications* (1970). Primary 10C05, 18F25, 20C05; Secondary 12A60, 16A40, 20G25, 57D65.

real subfield of the field of pth roots of unity, Γ its class group. It is known [5] that $\Gamma \cong \tilde{K}_0(\pi)$. Let ϕ denote the index in $\boldsymbol{Z}_p^\times$ of the subgroup generated by 2 and -1. Let Σ be the group (of signatures) having $\frac{1}{2}(p-1)$ components $\in \boldsymbol{Z}$, each divisible by 4, with sum divisible by 8.

$L_n(\pi^+) \cong L_n(\boldsymbol{Z}_2^+) \oplus L_n(p)$ *and* $L_n(\pi^-) \cong L_n(\boldsymbol{Z}_2^-) \oplus L_{n+2}(p)$, *where* $L_0(p) \cong \Gamma_2 \oplus \Sigma$, $L_1(p) \cong {}_2\Gamma$, $L_2(p) \cong \phi\boldsymbol{Z}_2$, $L_3(p)$ *has order* $2^{\phi+p-1}$ *and exponent* 2 *or* 4 *according as* $p \equiv \pm 1\,(4)$.

(iv) π *nonabelian of order* 8. We have the dihedral group D and the quaternion group Q. Distinguish the nonorientable versions of D by writing D^θ if for x of order 4, $w(x) = 1$ and D^- if $w(x) = -1$. In the table, $\boldsymbol{Z}$ denotes a signature with values divisible by 8.

	D^+	D^-	D^θ	Q^+	Q^-
L_0	$5\boldsymbol{Z}$	$\boldsymbol{Z} \oplus \boldsymbol{Z}_2$	$\boldsymbol{Z}_2$	$4\boldsymbol{Z}$	$\boldsymbol{Z}_2 \oplus \boldsymbol{Z}_2$
L_1	0	0	$\boldsymbol{Z}_2$	$2\boldsymbol{Z}_2$	0
L_2	$\boldsymbol{Z}_2$	$\boldsymbol{Z}_2$	$\boldsymbol{Z} \oplus \boldsymbol{Z}_2$	$\boldsymbol{Z} \oplus \boldsymbol{Z}_2$	$\boldsymbol{Z}_2$
L_3	$4\boldsymbol{Z}_2$	$\boldsymbol{Z}_2$	0	$4\boldsymbol{Z}_2$	$\boldsymbol{Z}_2$

The precise relation of these to the usual L groups depends on $K_1(\boldsymbol{Z}\pi)$. If $K_1(\boldsymbol{Z}\pi) \to K_1(\boldsymbol{Q}\pi)$ is injective, or has kernel of odd order, then $L_n(\pi) = L_n^s(\pi)$. For π abelian, the kernel (usually then denoted $SK_1(\boldsymbol{Z}\pi)$) is known to have odd order if the Sylow 2-subgroup of π is cyclic or a four group, and to be trivial if π is an elementary 2-group, or the direct sum of a cyclic 2-group with a group of order 2 [2, p. 624]. Consider, on the other hand, cases when $K_1(\boldsymbol{Z}\pi)$ is finite: this holds [2] if whenever $g, h \in \pi$ generate the same cyclic subgroup, h is conjugate to g or to g^{-1}. This is true in particular if π is abelian of exponent 2, 4 or 6 or nonabelian of order 6, 8 or 21 and in these cases we can easily check that $\mathrm{Wh}'(\pi) = 0$, so $L_n(\pi) = L_n^h(\pi)$.

A number of these groups had been computed previously. The discussion of known results in [9, §13A and §17E] should be augmented (at least) by the references [6] and [7]. Bak has recently announced (see [1] for a preliminary version) that $L_n^s(\pi) = L_n^h(\pi) = 0$ for n odd and π abelian of odd order. Also Bass has [3], [4] detailed results on $L_3^s(\pi)$ and $L_3^h(\pi)$ for π abelian, and $L_1(\pi)$ for π of exponent 2, obtained by very different methods.

Our results, particularly (iii), give explicit counterexamples to any over-naïve ideas about the structure of the $L_n(\pi)$, but nevertheless a fair regularity is apparent, particularly for L_2.

I now describe the outline of the proof. Let S be a semisimple algebra over $\boldsymbol{Q}$ (e.g., $\boldsymbol{Q}\pi$), R a $\boldsymbol{Z}$-order in S (e.g., $\boldsymbol{Z}\pi$), $\hat{R}$ its profinite completion, $\hat{S} = \hat{R} \otimes \boldsymbol{Q}$, $T = S \otimes \boldsymbol{R}$. One first shows, in the context of the L-theory of rings [**10**], that there is an exact sequence

$$\cdots \to L_i^X(R) \to L_i^X(\hat{R}) \oplus L_i^S(S) \to L_i^S(\hat{S}) \to L_{i-1}^X(R) \to \cdots$$

where the X signifies that determinants are all to be evaluated in $K_1(\hat{S})$. The construction of the boundary map $L_2^X(\hat{S}) \to L_1^X(R)$ and exactness here use linking forms; the rest of the proof follows the usual pattern of algebraic K-theory. Details will appear in [**11**].

Next, we compute $L_i^S(S)$. The map

$$L_i^S(S) \to L_i^S(\hat{S}) \oplus L_i^S(T)$$

is injective ('Hasse principle'): its cokernel $CL_i(S)$ is a sum of terms from simple components of S. For a component with centre K, the term is nonzero only if the involution is trivial on K, and is then given (possibly with dimensions shifted by 2) by $\boldsymbol{Z}_2$ $(i = 0)$, C_2 $(i = 1)$, ${}_2C$ $(i = 2)$, 0 $(i = 3)$, where C is the idèle class group of K. For cases (i), (ii), (iv) we only need to consider $K = \boldsymbol{Q}$, but, for (iii), $K = K_p$.

Now consider $\hat{R} = \prod_p \hat{R}_p$. Let $\bar{R}_p$ be the reduction of $\hat{R}_p$ modulo its radical. We say that $\hat{R}_p$ has good reduction if

$$\operatorname{Ker}(K_1\hat{R}_p \to K_1\hat{S}_p) \subset \operatorname{Ker}(K_1\hat{R}_p \to K_1\bar{R}_p).$$

The former kernel is always finite; the latter a profinite p-group. Using modular representation theory, one sees that, for all p, π, $\hat{\boldsymbol{Z}}_p\pi$ has good reduction. It follows for p odd, using a lifting theorem modulo the radical, that $L_i^X(\hat{R}_p) = L_i^S(\bar{R}_p)$, which is easy to compute.

We have an exact sequence

$$\cdots \to L_i^X(R) \to L_i^X(\hat{R}) \oplus L_i^S(T) \to CL_i(S) \to L_{i-1}^X(R) \to \cdots$$

and now verify, in each of the cases considered, that

$$\textstyle\prod_{p\,\mathrm{odd}} L_i^X(\hat{R}_p) \oplus \operatorname{Tors} L_i^S(T) \to CL_i(S)$$

is injective, and obtain the cokernel (which is finite). Indeed, the calculations here for (i) and the general case of (ii) both reduce to the case for $R = \boldsymbol{Z}$. It thus remains to compute $L_i^X(\hat{R}_2)$.

Now in the notation of [**10**],

$$L_i^K(\hat{\boldsymbol{Z}}_2\pi) \cong L_i^K(\overline{\boldsymbol{Z}_2\pi})$$

is a sum of copies of $\boldsymbol{Z}_2$, one for each irreducible 2-modular representation of π of type SPOT. We note in passing that these groups are easy to detect: for i even, by the Kervaire-Arf invariant, and for i odd, by Lee's semi-characteristics [**8**]. But only in (iii) does this yield anything essentially new. One can pass from L^K to L^X by an exact sequence (similar to one of Rothenberg)

$$\cdots \to L_i^X(\hat{Z}_2\pi) \to L_i^K(\hat{Z}_2\pi) \to \hat{H}^i(Z_2; V_2) \to \cdots$$

where V_2 is the image of $Nrd: K_1(\hat{Z}_2\pi) \to K_1(\hat{Q}_2\pi)$, and indeed of $(\hat{Z}_2\pi)^\times$.

The remainder—and it is the hardest part—of the calculation involves computing groups of units of $\hat{Z}_2\pi$. We need these in sufficient detail to calculate homomorphisms, as well as the terms in these sequences. I give two sample results of this kind.

π abelian, orientable case. $\{\pm g : g \in \pi\}$ *maps onto* $H^1(Z_2;(\hat{Z}_2\pi)^\times)$. Next, suppose π an elementary abelian 2-group with dual ρ. Each $\chi \in \rho$ gives a map $\hat{Q}_2\pi \to \hat{Q}_2$; the sum of these is an isomorphism. Now $\{a(\chi) : \chi \in \rho\}$ *comes from* $(\hat{Z}_2\pi)^\times$ *if and only if each* $a(\chi) \in \hat{Z}_2^\times$ *and, for each subgroup* H *of* ρ,

$$\prod\{a(\chi) : \chi \in H\} \equiv 1 \bmod |H|.$$

For π of odd order, $\hat{Z}_2\pi$ is an (unramified) maximal order; for π of order $2p$ we can split $\hat{Z}_2\pi$ into 2-blocks; the first one is $\hat{Z}_2[Z_2]$, and the rest have trivial defect group, hence are unramified.

One useful device to shorten some calculations is to use retractions. For example in the orientable case, $L_n(\pi) = L_n(1) \oplus \tilde{L}_n(\pi)$. For π of odd order, it is easier to follow the above chain of exact sequences for $\tilde{L}$, where nearly all the groups vanish.

For further calculations, one will need explicit invariants to detect elements in these groups. In many cases, the torsion subgroup of $L_n^X(R)$ maps injectively to $L_n^X(\hat{R}_2)$, and there is some hope of finding invariants, though $\hat{H}^0(Z_2; V_2)$ makes a numerically large and somewhat awkward contribution. For π orientable abelian the torsion in L_0, however, is annihilated by all invariants we know.

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