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APPELL POLYNOMIALS WHOSE GENERATING FUNCTION IS MEROMORPHIC ON ITS CIRCLE OF CONVERGENCE

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Let $\Phi(z) = \sum_0^\infty \beta_j z^j$ have radius of convergence r $(0 < r < \infty)$ and no singularities other than poles on the circle $|z| = r$. The Appell polynomials generated by Φ are given by

$$\pi_k(z) = \sum_{j=0}^{k} \beta_{k-j}\, z^j/j!, \qquad k = 0, 1, 2, \cdots.$$

An entire function g is said to possess a $\{\pi_k\}$ expansion if there is a complex sequence $\{h_k\}_0^\infty$ such that

$$\sum_{k=0}^{\infty} h_k \pi_k(z) \tag{1}$$

converges uniformly on compact sets to $g(z)$. In this note we show that the family of functions which have $\{\pi_k\}$ expansions is completely determined by the poles of Φ on $|z| = r$ together with the zeros of Φ in the closed disk $|z| \leqq r$.

Set $\Phi(z) = T(z)\phi_1(z)/P(z)$, where ϕ_1 is analytic and zero-free in $|z| \leqq r$ and T and P are polynomials whose zeros correspond respectively to the zeros of Φ in $|z| \leqq r$ and the poles of Φ on $|z| = r$. Let

$$P(z) = \prod_{q=1}^{\lambda} (1 - \alpha_q z)^{m(q)},$$

where $m(q)$ denotes the multiplicity of the pole α_q^{-1} of Φ, and let $m = \max m(q)$, $1 \leqq q \leqq \lambda$. It is relatively easy to characterize those complex sequences $\{h_k\}_0^\infty$ for which (1) converges. The following result was proved in [**2**], and can also be obtained as a special case of a theorem of W. T. Martin [**3**].

THEOREM A. *If $\{h_k\}_0^\infty$ is a complex sequence, then the following are equivalent*:

(i) *each of the series*

$$\sum_{k=0}^{\infty} \binom{k + m(q) - 1}{m(q) - 1} h_k \alpha_q^k, \qquad 1 \leqq q \leqq \lambda,$$

converges;

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(ii) *the series* (1) *converges for all z in some infinite bounded set;*

(iii) *the series* (1) *converges for all z, the convergence being uniform on every compact set.*

The problem of determining which entire functions g possess $\{\pi_k\}$ expansions is considerably more intricate, and the solution of this problem is our main result. Let

$$Q(z) = \prod_{q=1}^{\lambda} \{1 - \alpha_q z\}^{\min\{m(q), m-1\}},$$

let D denote the derivative operator, and let $\mathscr{F}$ denote the space of entire functions f such that

$$\lim_{n\to\infty} r^{-n}(D^n f)(0) = 0 \quad \text{and} \quad \lim_{n\to\infty} n^{m-1} r^{-n}(Q(D)D^n f)(0) = 0.$$

If $m = 1$ (Φ has only simple poles on $|z| = r$), the second condition reduces to the first, and $\mathscr{F}$ is the collection of all f such that $f^{(n)}(0) = o(r^n)$, $n \to \infty$. In general, $f^{(n)}(0) = o(r^n)$ is a necessary condition that f belong to $\mathscr{F}$, and the condition

$$f^{(n)}(0) = o(r^n/n^{m-1}), \qquad n \to \infty,$$

is sufficient. For each $k \geqq 0$, let L_k denote the linear functional given by

$$L_k(f) = \sum_{j=k}^{\infty} a_{j-k} f^{(j)}(0),$$

where $\sum a_j z^j$ is the power series for $T(z)/\Phi(z)$. It was shown in [**2**] that, if Φ is zero-free in $|z| \leqq r$, then g possesses a $\{\pi_k\}$ expansion if and only if g belongs to $\mathscr{F}$. The expansion in this case is unique, the coefficient sequence $\{h_k\}_0^\infty$ being given by $\{L_k(g)\}_0^\infty$ (provided one takes $T(z) \equiv 1$). There is an easy and beautiful extension of this result to the general case.

THEOREM B. *A necessary and sufficient condition that an entire function g possess a $\{\pi_k\}$ expansion is that the differential equation $T(D)f = g$ have a solution f which belongs to $\mathscr{F}$. If*

$$g(z) = \sum_{k=0}^{\infty} h_k \pi_k(z)$$

for all z, then there is an $f \in \mathscr{F}$ such that $T(D)f = g$ and $h_k = L_k(f)$, $k = 0, 1, 2, \cdots$. Conversely, if $f \in \mathscr{F}$ and $g = T(D)f$, then

$$g(z) = \sum_{k=0}^{\infty} L_k(f)\pi_k(z) \tag{2}$$

for all z, the convergence being uniform on every compact set.

PROOF. Let $\{p_k\}_0^\infty$ denote the Appell polynomial sequence generated by $\phi(z) = \Phi(z)/T(z)$; since ϕ is zero-free in $|z| \leqq r$, all the results obtained in [**2**] apply. Suppose that

$$g(z) = \sum_{k=0}^{\infty} h_k \pi_k(z) \tag{3}$$

for all z and set

$$f(z) = \sum_{k=0}^{\infty} h_k p_k(z). \tag{4}$$

It follows from Theorem A that the convergence of (3) is equivalent to that of (4), and is uniform on compact sets in both cases. Verify that $\pi_k = T(D)p_k$ and apply the operator $T(D)$ to both sides of (4). This yields

$$(T(D)f)(z) = \sum_{k=0}^{\infty} h_k \pi_k(z) = g(z).$$

From (4) and the remark preceding Theorem B, it follows that $f \in \mathscr{F}$ and that $h_k = L_k(f)$, $k = 0, 1, 2, \cdots$.

Suppose now that $f \in \mathscr{F}$ and $g = T(D)f$. From the remark preceding Theorem B, we have

$$f(z) = \sum_{k=0}^{\infty} L_k(f) p_k(z). \tag{5}$$

Applying $T(D)$ to both sides of (5), we obtain (2), and this completes the proof.

Unless Φ is zero-free in $|z| < r$, the $\{\pi_k\}$ expansions are not unique. Let $\mathscr{H}$ denote the space of all sequences $\{h_k\}_0^\infty$ such that

$$\sum_{k=0}^{\infty} h_k \pi_k(z) = 0$$

for all z (equivalently, for all z in some infinite bounded set). Set $T(z) = T_0(z)T_1(z)$, where T_0 is a polynomial with no zero outside the disk $|z| < r$ and T_1 is a polynomial with no zero off the circle $|z| = r$. Let $\mathscr{H}_0$ denote the space of all sequences $\{h_k\}_0^\infty$ such that

$$u(z) = \sum_{k=0}^{\infty} h_k z^k/k!$$

satisfies the differential equation $T_0(D)u = 0$. It is known (and easy to prove) [**1**, p. 25] that $\mathscr{H}_0 \subseteq \mathscr{H}$. We shall prove that $\mathscr{H} = \mathscr{H}_0$ by showing that the dimension of $\mathscr{H}$ does not exceed the degree of T_0, which is the dimension of $\mathscr{H}_0$. This approach is necessary since our technique leads to a somewhat different characterization of $\mathscr{H}$.

THEOREM C. $\mathscr{H} = \mathscr{H}_0$.

PROOF. Suppose $\{h_k\}_0^\infty$ belongs to $\mathscr{H}$. It follows from the argument used to prove Theorem B that the function $f(z) = \sum_{k=0}^\infty h_k p_k(z)$ belongs to $\mathscr{F}$ and satisfies $T(D)f = 0$. Set $F = T_0(D)f$. Then

$$0 = T(D)f = T_1(D)\{T_0(D)f\} = T_1(D)F.$$

Since $f \in \mathscr{F}$, it follows that $F \in \mathscr{F}$; therefore F satisfies

$$F^{(n)}(0) = o(r^n), \qquad n \to \infty. \tag{6}$$

The solutions of $T_1(D)F = 0$ are well known, and the only one which satisfies (6) is $F \equiv 0$. Therefore $T_0(D)f = 0$. The dimension of the solution space of $T_0(D)f = 0$ is equal to the degree of T_0, and to complete the proof, we need only show that the linear mapping which takes the sequence $\{h_k\}_0^\infty$ in $\mathscr{H}$ onto the function $\sum h_k p_k(z)$ is 1–1. This is equivalent to showing that $\sum h_k p_k(z) = 0$ for all z implies $h_k = 0$, $k = 0, 1, 2, \cdots$. This was established in [**2**]; therefore the proof is complete.

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