ON THE MULTIPLICITY OF THE SPECTRUM OF THE SPACE OF CUSP FORMS OF GL_n[1]

BY J. A. SHALIKA

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In this note I wish to announce certain preliminary results concerning a class of special functions arising in the theory of representations of groups defined over local fields. Before the preparation of this manuscript, the author obtained a copy of the recent paper [4] of I. M. Gelfand and D. A. Kajdan. Several of the results announced here are proved in their paper. I hope to indicate below what I have obtained independently and also what new results I have obtained since the paper of Gelfand and Kajdan became available. I wish to thank A. W. Knapp for the proof of a central lemma stated below.

1. **A conjecture.** Let k be a global field. Let G be an algebraic group defined over k. If R is a commutative k-algebra with identity, G_R will denote the group of points of G rational over R. The notation $G(R)$ will also be used when convenient. $R^\times$ will denote the unit group of R. Let A denote the ring of adeles of k. Let ω be a (unitary) character of $A^\times$ trivial on $k^\times$. For $G = GL_n$, let

$$C_\omega = {}^0L_2(G_A/G_k, \omega)$$

denote the space of cusp forms on G_A associated with ω. Let $\hat{G}_A$ denote the set of equivalence classes of admissible, irreducible, unitary representations of G_A. For $G = GL_n$, $\Pi \in \hat{G}_A$, let $m_0(\Pi, \omega)$ denote the multiplicity with which Π occurs in C_ω.[2]

In this note, I want to present some evidence in support of the following:

CONJECTURE 1. For $\Pi \in \widehat{GL_n}(A)$, $m_0(\Pi, \omega) \leq 1$.

The conjecture has, of course, been proved for GL_2. Classically this statement, somewhat reformulated, appears in the well-known works of E. Hecke and H. Maass in their development of the theory of automorphic forms on GL_2. A systematic and general approach to Hecke theory was carried out in the framework of the theory of representations by H. Jacquet and R. .P. Langlands [11]. The theory was also developed in a general setting by A. Weil [18]. In the work of Jacquet-Langlands, the proof of the simplicity of the spectrum of the space of cusp forms for GL_2 is reduced,

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[2] For $ch(k) = 0$, it is well known [11] that the assumption $m_0(\Pi, \omega) > 0$ implies that Π is admissible.

starting with the Fourier series of such a form, to the uniqueness of the "local Whittaker model" for the various representations of GL_2 over a local field.

In 1962, the uniqueness of, what perhaps should now be called, a Whittaker model for the representations of GL_n (more generally Chevalley groups) over a finite field was formulated and proved by I. M. Gelfand and M. I. Graev [2]. Using these results, S. I. Gelfand proved the existence of such a model for the super-cuspidal representations of GL_n over a finite field [5].

While attempting to generalize the results of [2] and [5] to local fields, I observed the existence of a natural "Fourier series" for cusp forms on GL_n. The consideration of such a series was largely motivated by Proposition 3 of [5].

2. **The Fourier series of a cusp form.** Let $N = N_n$ denote the subgroup of super-triangular matrices of GL_n. Let χ be a one-dimensional unitary representation of N_A such that

(1) χ is trivial on N_k,

(2) χ is "nondegenerate", i.e., the restriction of χ to a simple root group is nontrivial.

Let v be a place of k. Let k_v denote the completion of k at v.

Let

$$G_0 = \prod_{v \text{ finite}} GL_n(k_v)$$

(restricted product) and

$$G_\infty = \prod_{v \text{ infinite}} GL_n(k_v).$$

Let $C^\infty(G_0)$ denote the space of locally constant, complex-valued functions on G_0 and $C^\infty(G_\infty)$ the space of infinitely differentiable functions on G_∞. Let $P = P_n$ denote the subgroup of GL_n consisting of matrices of the form (x_{ij}) where $x_{n1} = \cdots = x_{n\,n-1} = 0$. Let P'_n denote the subgroup of P_n defined by the additional restriction $x_{nn} = 1$. For $G = GL_n$, let $^0C^\infty(G_A/P_k)$ denote the space of $\phi \in C^\infty(G_A)$ such that

(3) $\phi(xp) = \phi(x) (x \in G_A, p \in P_k)$

and satisfying the additional hypothesis:

(4) if U is the unipotent radical of a proper parabolic subgroup of GL_n defined over k, then

$$\int_{U_A/U_k} \phi(xu)\, du^* = 0.$$

(Here du^* is an invariant measure on the homogeneous space U_A/U_k.)

For $\gamma \in GL_r(A)$, let

$$d[\gamma] = \begin{pmatrix} \gamma & 0 \\ 0 & 1_{n-r} \end{pmatrix}$$

where 1_{n-r} is the $(n - r) \times (n - r)$ identity matrix.

Finally, for $\phi \in {}^0 C^\infty(G_A/P_k)$, χ as above, let

$$W_\phi(x) = \int_{N_A/N_k} \phi(xu)\chi(u)\,du^* \qquad (x \in G_A).$$

Then we have

THEOREM 1. *Let* $\phi \in {}^0 C^\infty(G_A/P_k)$. *Then*

$$\phi(1) = \sum_{\gamma_{n-1} \in GL_{n-1,k}/P'_{n-1,k}} \cdots \sum_{\gamma_1 \in GL_{1,k}} W_\phi(d[\gamma_{n-1}] \cdots d[\gamma_1])$$

$$= \sum_{\gamma \in GL_{n-1,k}/N_{n-1,k}} W_\phi(d[\gamma]).$$

COROLLARY. *The map* $\phi \to W_\phi$ *from* ${}^0 C^\infty(G_A/P_k)$ *into the space of complex-valued functions on* G_A *is injective.*

3. **Reduction to the local case.** From the results of §1, following Jacquet-Langlands, one may readily reduce Conjecture 1 to the uniqueness of a "local Whittaker model". In positive characteristic, this is immediate. In characteristic zero, there are some convergence problems. For this reason, we introduce an appropriate space of distributions. The definition of this space is well-adapted to the infinitesimal description of B. Kostant of class one representations of real Lie groups.

For $G = GL_n$, let $C_c^\infty(G_A)$ denote the set of elements of $C^\infty(G_A)$ with compact support. Let $\Pi \in \hat{G}_A$. Let H denote the corresponding Hilbert space, with inner product defined by the norm $\| \ \|$. Let $\mathscr{D}(\Pi)$ denote the linear subspace of H spanned by elements of the form $\Pi(f)v$ where $f \in C_c^\infty(G_A)$, $v \in H$. Let $\mathfrak{G}$ be the universal enveloping algebra of G_∞. $\mathfrak{G}$ acts on $C_c^\infty(G_\infty)$ (on the left) as differential operators. $\mathfrak{G}$ acts on $\mathscr{D}(\Pi)$ in such a way that $\mathfrak{G}$ commutes with the natural action of $C_c^\infty(G_0)$ and so that

$$\Pi(D \circ f)v = (-1)\Pi(D)\Pi(f)v$$

for $D \in \mathrm{Lie}(G_\infty)$, $f \in C_c^\infty(G_\infty)$. Let K_0 be the maximal compact subgroup of G_0 defined by the canonical integral structure on GL_n. Let σ be a finite-dimensional (unitary) representation of K_0. Let E_σ denote the corresponding projection operator on H. Then

$$E_\sigma \mathscr{D}(\Pi) = \mathscr{D}_\sigma(\Pi) \subset \mathscr{D}(\Pi).$$

We topologize $\mathcal{D}_\sigma(\Pi)$ by the seminorms

$$\|v\|_D = \|Dv\| \qquad (D \in \mathfrak{G}, \; v \in \mathcal{D}(\Pi)).$$

This defines a unique topology on $\mathcal{D}(\Pi) = \bigcup_\sigma \mathcal{D}_\sigma(\Pi)$. Let K_∞ be a fixed maximal compact subgroup of G_∞. Put $K = K_0 \cdot K_\infty$. Let H_K denote the space of K-finite vectors in H. Then

LEMMA 1. $H_K \subset \mathcal{D}(\Pi)$ and is dense in $\mathcal{D}(\Pi)$.

The proof of Lemma 1 is reduced to the case of a connected Lie group by using the theory of semidirect products of G. W. Mackey. The proof then follows immediately from [**7**, p. 9].

Now let $\mathcal{D}'(\Pi)$ denote the topological dual space of $\mathcal{D}(\Pi)$. Let I be an isometry of H into $L_2(G_A/G_k, \omega)$ commuting with the action of G_A. Then I maps $\mathcal{D}(\Pi)$ into $C^\infty(G_A)$. For $v \in \mathcal{D}(\Pi)$, let $\mathcal{A}(v) = I(v)(1)$. Then

PROPOSITION 1. $\mathcal{A} \in \mathcal{D}'(\Pi)$.

Let G_A act on $\mathcal{D}'(\Pi)$ by duality. Let $\mathcal{D}'_\chi(\Pi)$ denote the space of $\lambda \in \mathcal{D}'_\chi(\Pi)$ such that

$$u \circ \lambda = \chi^{-1}(u)\lambda \qquad (u \in N_A).$$

Define $\lambda_{\mathcal{A}}$ by

$$\lambda_{\mathcal{A}}(v) = W_{I(v)}(1) \qquad (v \in \mathcal{D}(\Pi)).$$

COROLLARY. $\lambda_{\mathcal{A}} \in \mathcal{D}'_\chi(\Pi)$.

As in [**11**], one can prove that *Conjecture 1 is implied by*
CONJECTURE 2. For χ nondegenerate, $\Pi \in \widehat{GL}_n(A)$, Dim $\mathcal{D}'_\chi(\Pi) \leq 1$.

4. **The local Whittaker model.** Let $G_v = GL_n(k_v)$. Let $\hat{G}_v$ denote the set of admissible, irreducible, unitary representations of G_v. For $\Pi \in \hat{G}_v$; one can define $\mathcal{D}_v(\Pi)$ and $\mathcal{D}'_{\chi,v}(\Pi)$ as above. For $\Pi \in \hat{G}_A$, write Π uniquely in the form $\bigotimes_v \Pi_v$ ($\Pi_v \in \hat{G}_v$). We have

LEMMA 2. $\bigotimes_v \mathcal{D}_v(\Pi_v)$ is dense in $\mathcal{D}(\Pi)$.

(Alternatively, one could define $\mathcal{D}(\Pi)$ by this lemma.)

There are two consequences. Firstly,

PROPOSITION 2. Let $\Pi \in \widehat{GL}_n(A)$. Suppose $m_0(\Pi, \omega) > 0$. Then, for all places v of k, Dim $\mathcal{D}'_{\chi,v}(\Pi_v) > 0$.

Secondly, one sees immediately that *Conjecture 2 is implied by*
CONJECTURE 3. For all places v of k and all $\Pi \in \hat{G}_v$, Dim $\mathcal{D}'_{\chi,v}(\Pi) \leq 1$.
Conjecture (3) has been proved by Gelfand and Kajdan (Theorem 4 of [**4**]) for nonarchimedean local fields. Hence, if $ch(k) > 0$, it follows,

immediately from the result of Gelfand and Kajdan and §1 above, that we have the

THEOREM. *Suppose the characteristic of k is positive. Let Π belong to $\widehat{GL_n}(A)$. Then $m_0(\Pi, \omega) \leqq 1$.*

In characteristic zero, again by using the results of Gelfand and Kajdan and §1 and §2 above, Conjecture 1 is immediately reduced to Conjecture 3 for $GL_n(R)$ and $GL_n(C)$. At the present writing, Conjecture 3 is still open for these groups other than for class one representations (see below).

It is of some interest to consider the analogue of Conjecture 3 for a connected, reductive group G defined over k_v as well as for connected reductive Lie groups. The conjecture was indeed proved by Gelfand and Kajdan for a suitably restricted class of Chevalley groups over non-archimedean local fields. For v nonarchimedean, let K_v be a "good" maximal compact subgroup of G_{k_v} as defined in [16]. In the direction of Conjecture 3 in the general case, one has

PROPOSITION 3. *Let G be a connected reductive group and Q be a minimal parabolic subgroup of G both defined over k_v. Suppose v is nonarchimedean. Let K_v be a maximal compact subgroup of $G(k_v)$ as above. Let $\Pi \in \hat{G}(k_v)$. Suppose Π has a K_v fixed vector. Then there exists a (quasi-) character α of $Q(k_v)$ such that Π_v is a quotient of $\operatorname{ind}_{\phi(k_v)\uparrow G(k_v)}\alpha$.*[3]

An analogous result for connected semisimple Lie groups was obtained by B. Kostant [13].

For Π as in Proposition 3, the fact that $\operatorname{Dim} \mathscr{D}'_{\chi,v}(\Pi) \leqq 1$ is a simple consequence of the method of F. Bruhat [1]. We also have

PROPOSITION 4. *Let G_1 be a connected semisimple real Lie group, K_1 a maximal compact subgroup. Let Π be an irreducible unitary representation of G_1 having a K_1-fixed vector. Then $\operatorname{Dim} \mathscr{D}'_\chi(\Pi) \leqq 1$.*

The proof of Proposition 4 may be obtained as a consequence of a theorem of F. Bruhat and the following lemma whose proof was kindly communicated to me by A. W. Knapp.

LEMMA 3. *Let Π be a representation of the class one principal series of a connected semisimple Lie group G_1. Π acts on $C_c^\infty(K_1)$. Let B be a formally skew-Hermitian, semidefinite, bilinear form defined on the K_1-finite vectors in $C_c^\infty(K_1)$. Then B is continuous in the Schwartz topology.*

The proof of this lemma relies on [12], [14] and [17].

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[3] Since the preparation of this manuscript, the author has been informed that generalizations of Proposition 3 had been obtained independently by W. Casselman, H. Jacquet and Harish-Chandra.

As a consequence of Propositions 3 and 4, one obtains explicit integral representations for the Whittaker functions associated with the class one representations of the groups described in those propositions. This constitutes a step in the direction of explicitly determining the Fourier coefficients of cusp forms attached to GL_n. These integrals[4] were studied by H. Jacquet in [10].

Now suppose v is archimedean. As a consequence of Proposition 4, by using the theory of semidirect products of Mackey, one can prove that if Π is an irreducible class one unitary representation of $GL_n(k_v)$ then Dim $\mathscr{D}_{\chi,v}(\Pi) \leqq 1$. This result together with Proposition 3 implies

THEOREM 2. *Suppose* $\Pi \in \widehat{GL_n}(A)$ *is of class one with respect to* K. *Then* $m_0(\Pi, \omega) \leqq 1$.

As indicated above, Theorem 2 may be deduced from the work of Gelfand and Kajdan in positive characteristics.

5. **Super-cuspidal representations of** GL_n. In this section v will denote a nonarchimedean place of k. Let G_v denote $GL_n(k_v)$.

The following theorem was proved independently by the present author and by Gelfand and Kajdan.

THEOREM 3. *Suppose* Π *is a super-cuspidal representation of* G_v. *Then* Dim $\mathscr{D}'_{\chi,v}(\Pi) = 1$.

This result is partially a consequence of the following L_2 analogue of Theorem 4 of [4].

PROPOSITION 5. *Let* χ *be a nondegenerate one-dimensional unitary representation of* N_{k_v}. *Then the commuting algebra (in the* L_2 *sense) of the representation* $\mathrm{ind}_{N_{k_v} \uparrow G_v} \chi$ *is commutative.*

I have also proved a C_c^∞ analogue of Proposition 5. This result does not in any way imply Conjecture 3 for nonarchimedean fields. However, one can deduce that Dim $\mathscr{D}'_{\chi,v}(\Pi) \leqq 1$ for Π super-cuspidal. The fact that, in this case, a Whittaker model actually exists was proved by the author by essentially the same method as the proof of Theorem 1. By similar methods, I have also proved that, for all $\Pi \in \hat{G}_v$, there exists a unitary character χ_0 of N_{k_v} (not necessarily nondegenerate) such that Dim $\mathscr{D}'_{\chi_0,v}(\Pi) \geqq 1$.

Now, for $f \in C_c^\infty(G_v)$ and $x \in G_v$, define $L_x f$ and $R_x f$ so that

$$(L_x f)(y) = f(x^{-1}y),$$

$$(R_x f)(y) = f(yx) \qquad (y \in G_v).$$

As in [2], [4], define an involution θ on G_v by

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[4] See also the references given in [10].

$$\theta(x) = w_0 \, {}^t x w_0.$$

Here w_0 denotes the $n \times n$ matrix with 1 along the antidiagonal and zero elsewhere. ${}^t x$ denotes the transpose of the $n \times n$ matrix x. Proposition 5 is a consequence of

PROPOSITION 6. *Let T be a distribution on G_v satisfying $T(L_{u_1^-} f R_{u_2})$
$= \chi(u_1 u_2) T(f)$ for all $f \in C_c^\infty(G_v)$, $u_1, u_2 \in N_{k_v}$. Then*

$$T(f^\theta) = T(f)$$

for all $f \in C_c^\infty(G_v)$.

This proposition was proved independently by the present author and by Gelfand and Kajdan.

Since the writing of the first draft of this manuscript, the author succeeded in proving the analogue of Proposition 6 for v archimedean, provided T is a measure. At the present writing, I have not been able to extend this latter result to a general distribution. However, I have been able to show, by using the method of proof of Theorem 4 of [4], that the analogue of Proposition 6 for v archimedean and general T implies Conjecture 1.[5]

6. **L-functions.** Proceeding by analogy with the definition of the Gauss sums defined by Gelfand and Kajdan, I have defined a natural L-function associated with a pair of cusp forms associated to *class one* representations of the groups GL_n and GL_{n-1} respectively. These L functions are defined by everywhere convergent integrals as in the work of Jacquet-Langlands. The proof of the functional equation for these L-functions may be obtained by analogy with the method of the paper of Gelfand and Kajdan. By using Theorem 1 of the present paper, one can show that these L-functions are given by Euler products, provided these products converge. As in Jacquet-Langlands, the local Euler factors may be expressed as non-explicit integrals involving pairs of Whittaker functions. It is reasonable to expect that explicit expressions for these Euler factors (in the class one case) can be obtained by a further study of Whittaker functions. I hope that Propositions 3 and 4 above will be useful in this direction.

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[5] Since the preparation of this manuscript, I have succeeded in proving Proposition 6 for v archimedean, provided T is an eigendistribution for the Casimir operator associated with G_v. This result is sufficient to prove the conjectures of this paper.

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DEPARTMENT OF MATHEMATICS, THE JOHNS HOPKINS UNIVERSITY, BALTIMORE, MARYLAND 21218