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EUCLIDEAN SUBRINGS OF GLOBAL FIELDS[1]

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1. **Introduction.** The purpose of this note is to announce some results regarding the existence of euclidean subrings of global fields.

We first state the problem and give its history. Let F be a global field. So F is a finite extension of the rational numbers Q or F is a function field of one variable over a finite field k, where k is algebraically closed in F. Let S be a finite nonempty set of prime divisors of F such that S includes all infinite (i.e., archimedean) prime divisors. If P is a finite (i.e., nonarchimedean) prime divisor we denote by O_P its valuation ring in F. Now, given a finite set S of the above type, we get a ring

$$O_S = \bigcap_{P \notin S} O_P$$

where P ranges over all prime divisors of F. We note in particular that if F is a number field and S the set of infinite prime divisors of F then O_S is just the ring of F-integers.

It is easy to see that there always exists a finite set S satisfying the above hypothesis such that O_S is a unique factorization domain. Hence it seems natural to ask the following two questions:

I. Does there always exist an S such that O_S is a euclidean ring?

II. Can one find an algorithm on O_S for suitably chosen S which is related in some way to the arithmetic of the field F?

The history of the above two questions is as follows: In a series of articles [**1**]–[**4**] Armitage discussed I and II for function fields over arbitrary ground fields. He insisted on a choice of algorithm related to the norm from F to a rational subfield. He showed that if the ground field of F is infinite, then an algorithm of his spacial type was possible if and only if the genus of F is zero. He also discussed the case when the ground field of F is finite, but again the only situation in which he gave a positive answer to I was when F is of genus zero. In [**6**], Samuel also discussed I for function fields F with arbitrary fields of constants, but here also he did not get above genus zero. Finally, in [**5**], M. Madan and the present author showed that the answer to both I and II is yes for function fields of genus one over finite fields. The method in [**5**] was to specifically construct an S and an algorithm on O_S for given F.

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[1] ADDED IN PROOF. After this announcement went to press, the author discovered that Theorem 2 was proved by O. T. O'Meara in *On the finite generation of linear groups over Hasse domains*, J. Reine Angew. Math. **217** (1965), 79–108. MR **31** #3513.

In the next section we indicate a proof that the answer to both I and II is yes for arbitrary global fields F. Full details of the proof and applications of the results will appear elsewhere.

2. **Results.** Let F be a global field. If P is a finite prime divisor of F we denote by $N(P)$ the absolute norm of P and we associate with each such P a normalized valuation $|\ |_P$ as follows: $|O|_P = 0$ and if $x \in F - \{0\} = F^*$, then $|x|_P = N(P)^{-n}$, where P^n is the power to which P appears in the principal divisor (x) determined by x in F. Now if P is an infinite prime divisor then P corresponds to an embedding σ_P of F into the complex numbers and we determine a normalized valuation $|\ |_P$ associated to P in the following way: If $\sigma_P(F)$ is a subfield of the real numbers then $|x|_P = |\sigma_P(x)|$ for all $x \in F$, where $|\ |$ is the ordinary real absolute value. Finally if $\sigma_P(F)$ is not a subfield of the reals, we set $|x|_P = |\sigma_P(x)|^2$ for all $x \in F$, where $|\ |$ is the usual complex absolute value. Hence letting P range over all prime divisors of F, we have the well-known formula

$$\prod_P |x|_P = 1 \tag{1}$$

for all $x \in F^*$.

If P is a prime divisor of F we denote by F_P the completion of F with respect to the valuation $|\ |_P$. These fields F_P are all locally compact and if P is finite we denote by R_P the maximal compact subring of F_P. We call the restricted topological product of the F_P with respect to the R_P the ring of adeles of F and denote it by F_A. We further identify F with its diagonal embedding in F_A.

Now if F is a number field we denote by S_∞ the set of infinite prime divisors of F and if F is a function field over a finite field we fix a prime divisor P_∞ of F and set $S_\infty = \{P_\infty\}$. Next if $x \in F^*$ we set

$$V(x) = \{\zeta \in F_A \mid |\zeta_P|_P < |x|_P \text{ for } P \in S_\infty \text{ and } |\zeta_P|_P \leqq |x|_P \text{ for } P \notin S_\infty\}.$$

THEOREM 1.

$$F_A = \bigcup_{x \in F^*} (V(x) + F).$$

INDICATION OF PROOF. If F is a function field and k its exact field of constants we use the Riemann-Roch theorem to choose $t \in F$ such that $F/k(t)$ is a separable extension and $|t|_{P_\infty} > 1$, with $|t|_P \leqq 1$ for all $P \neq P_\infty$. If F is a number field we let H denote the field of real numbers and otherwise H will denote $k((t^{-1}))$, where $k((t^{-1}))$ is the quotient field of the ring of formal power series in t^{-1} over k. Next we set $F_\infty = F \otimes_L H$, where $L = Q$ if F is a number field and $L = k(t)$ otherwise. Viewing F_∞ as a topological algebra over H we identify it with the subalgebra of F_A, $\prod_{P \in S_\infty} F_P$.

Setting $X = F_\infty \times \prod_{P \notin S_\infty} R_P$, we observe that $F_A = X + F$ (see [7]). Let $\{\omega_1, \ldots, \omega_n\}$ be an integral basis of F over L with respect to Γ, where Γ is the ring of rational integers if F is a number field and otherwise $\Gamma = k[t]$. Finally we show that if $\zeta \in X$, then there exist $q, p_1, \ldots, p_n \in \Gamma$ such that $q \neq 0$ and $q\zeta - (p_1\omega_1 + \cdots + p_n\omega_n)$ has the property that

$$|(q\zeta - (p_1\omega_1 + \cdots + p_n\omega_n))_P|_P < 1 \quad \text{for } P \in S_\infty$$

and

$$|(q\zeta - (p_1\omega_1 + \cdots + p_n\omega_n))_P|_P \leqq 1 \quad \text{for } P \notin S_\infty,$$

i.e., $\zeta \in V(q^{-1}) + F$. Q.E.D.

Let S be a finite set of prime divisors of F such that $S \supseteq S_\infty$. We define a function φ_S from F to the nonnegative real numbers given by $\varphi_S(x) = \prod_{P \in S} |x|_P$. We note that, in view of (1), φ_S is integral valued when restricted to O_S. Further in the case when F is a number field and $S = S_\infty$, then, for all $x \in F$, $\varphi_S(x) = |N_{F/Q}(x)|$. Also when F is a function field, then for any choice of $S \supseteq S_\infty$, there exist $y \in F - k$ such that O_S is the integral closure of $k[y]$ in F and, for all $x \in F$, $\varphi_S(x) = |N_{F/k(y)}(x)|_\infty$, where $|\ |_\infty$ is the valuation associated to the pole divisor of y in $k(y)$ and normalized as above.

THEOREM 2. *There exists a finite set S of prime divisors of F such that $S \supseteq S_\infty$ and O_S is euclidean with respect to the map φ_S.*

INDICATION OF PROOF. By Theorem 1, $F_A = \bigcup_{x \in F^*} (V(x) + F)$. Now by compactness of F_A/F (see [7]) and the fact that $V(x)$ is open in F_A for every $x \in F$, there exist $x_1, \ldots, x_r \in F^*$ such that

$$F_A = \bigcup_{i=1}^{r} (V(x_i) + F).$$

Finally we show that if $S = \{P | P \in S_\infty$ or there exist $i_0, 1 \leqq i_0 \leqq r$ such that $|x_{i_0}|_P \neq 1\}$, then S is a finite set, $S \supseteq S_\infty$ and O_S is euclidean with respect to φ_S.

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