BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 79, Number 2, March 1973

NORMAL FORMS FOR SYSTEMS OF SEMILINEAR HYPERBOLIC PARTIAL DIFFERENTIAL EQUATIONS

BY J. EISENFELD AND P. M. ZARETZKI

Communicated by Hans F. Weinberger, September 13, 1972

1. **Introduction.** That a second order semilinear hyperbolic partial differential equation in two independent variables can be transformed into normal form so that differentiation is with respect to characteristic directions is well known. In this note we announce a generalization of this idea to systems of second-order semilinear hyperbolic equation in two variables. As an intermediate step a variational principle for the eigenvalues of a strongly-damped eigenparameter problem is developed [**1**]. Details will appear elsewhere.

2. **Definitions and notation.** (a) As usual an $n \times n$ Hermitian matrix P is positive definite if and only if the eigenvalues of P are strictly positive. We write $P > 0$.

(b) We define an inner product on C^n by

$$(x, y) = \sum_{i,j=1}^{n} x_i P_{ij} \bar{y}_j. \tag{2.1}$$

(c) If A, B, C are $n \times n$ matrices, define quadratic forms a, b, c by

$$a(x) = (Ax, x), \quad b(x) = (Bx, x), \quad c(x) = (Cx, x). \tag{2.2}$$

(d) The quadratic eigenparameter equation

$$L(\lambda)x = \lambda^2 Ax + \lambda Bx + Cx = 0 \tag{2.3}$$

is called strongly damped if

$$\begin{aligned} &\text{(i)} \quad PA > 0, \qquad \text{(ii)} \quad a, b, c \text{ are real valued,} \\ &\text{(iii)} \quad b^2(x) > 4a(x)c(x) \quad \text{for } x \neq 0. \end{aligned} \tag{2.4}$$

(e) The first order semilinear system of partial differential equations

$$v_t = Dv_x + h(x, t, v) \tag{2.5}$$

is in normal form if D is a diagonal matrix.

(f) A (single-valued) real valued function f defined on a real interval I is real analytic on I if and only if f has a convergent Taylor's series expansion about each point of I always with nontrivial interval of convergence.

AMS(*MOS*) *subject classifications* (1970). Primary 35L10, 35L40; Secondary 15A18.

3. **Main results.**

THEOREM 1. ([**1**]–[**4**]). *Let $P_+(x)$, $P_-(x)$ denote the two roots of $(L(\lambda)x, x) = 0$ with $P_+(x) > P_-(x)$. Then the strongly damped equation* (2.2) *has n primary eigenvalues*

$$\lambda_1^+ > \lambda_2^+ > \cdots > \lambda_n^+ \tag{3.1i}$$

and n secondary eigenvalues

$$\lambda_1^- \leqq \lambda_2^- < \cdots \leqq \lambda_n^- \tag{3.1ii}$$

and these eigenvalues are given by the variational principles

$$\lambda_k^+ = \min_{\dim V = k} \max_{x \in V} P_+(x) = P_+(\Phi_k^+) \tag{3.2i}$$

and

$$\lambda_k^- = \max_{\dim V = k} \min_{x \in V} P_-(x) = P_-(\Phi_k^-), \tag{3.2ii}$$

where $\Phi_k^\pm$ are the corresponding eigenvectors:

$$L(\lambda_k^\pm)\,\Phi_k^\pm = 0. \tag{3.3}$$

In the following two theorems we suppose that the $n \times n$ matrices A, B, C and P depend on x and t only through real analytic dependence on an intermediate variable $\eta = \eta(x, t)$; it is required that η admit first partial derivatives with respect to x and t.

THEOREM 2. *Let M be the $2n \times 2n$ matrix*

$$M = \begin{pmatrix} E & F \\ A^{-1}G & A^{-1}H \end{pmatrix} \tag{3.4}$$

where A, E, F, G, H are $n \times n$ matrices with A and F nonsingular. Let

$$B = -A(E + FA^{-1}HF^{-1}), \qquad C = AF(A^{-1}HF^{-1}E - A^{-1}G). \tag{3.5}$$

Let A, B, C satisfy (2.4). *Then the first order system of partial differential equations*

$$w_t = Mw_x + g(x, t, w) \tag{3.6}$$

is reducible to the normal form (2.5) *where the diagonal components of D are the eigenvalues of* (2.3).

THEOREM 3. *The system of second order partial differential equations*

$$Au_{tt} + Bu_{xt} + Cu_{xx} = f(x, t, u, u_x, u_t) \tag{3.7}$$

is equivalent to the first order system (3.6) *where*

$$w = u_x \oplus u_t, \tag{3.8}$$

$$M = \begin{pmatrix} O_n & I_n \\ -A^{-1}C & -A^{-1}B \end{pmatrix}. \tag{3.9}$$

The system (3.6) is reducible to the normal form (2.5) where the diagonal components of D are computed by the variational principle (3.2).

4. **Remarks.** A counterexample shows the importance of the intermediate variable η in Theorem 2 and Theorem 3. Another counterexample shows that the analytic dependence of M on η cannot, in general, be relaxed to C^∞ dependence. A final counterexample shows that condition (2.4iii) cannot be replaced by weak inequality.

References

1. R. J. Duffin, *A minimax theory for overdamped networks*, J. Rational Mech. Anal. **4** (1955), 221–233. MR **16**, 979.

2. ———, *The Rayleigh-Ritz method for dissipative or gyroscopic systems*, Quart. Appl. Math. **18** (1960/61), 215–221. MR **22** #12775.

3. J. Eisenfeld, *On symmetrization and roots of quadratic eigenvalue problems*, J. Functional Anal. **9** (1972), 410–422.

4. ———, *Block diagonalization of partitioned matrix operators*, J. Linear Algebra Appl. (to appear).

Department of Mathematics, Rensselaer Polytechnic Institute, Troy, New York 12181

Current address (J. Eisenfeld): Department of Mathematics, University of Texas at Arlington, Arlington, Texas 76010