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# FUNCTIONS WITH A SPECTRAL GAP

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**Introduction.** In harmonic analysis, it is important to know how various properties of a function on $\boldsymbol{R}^n$ reflect themselves as restrictions on its *spectrum*, i.e., the support of its (distributional) Fourier transform. Thus, according to Paley and Wiener, a compact spectrum is characteristic of entire functions of exponential type. In this note we consider a milder restriction: it is only required of the spectrum that it be smaller than the whole $n$-space. Our results extend those of Levinson, Logan, Ehrenpreis and Malliavin; cf. also Boas [**1**]. Here we give only bare outlines of proofs; we employ standard vector notations: $t = (t_1, \ldots, t_n)$ and $x = (x_1, \ldots x_n)$ are points of $\boldsymbol{R}^n$ and $(t, x)$ denotes $\sum_1^n t_j x_j$; $|t| = (t, t)^{1/2}$, and $dt$ denotes Haar measure on $\boldsymbol{R}^n$.

1. A *gap* in a distribution on $\boldsymbol{R}^n$ is a nonvoid open ball disjoint from its support. A *spectral gap* in a tempered distribution is a gap in its Fourier transform. In particular, an $L^1$ function $f$ has a spectral gap if its Fourier transform $\hat{f}(x)$ vanishes on some nonvoid open set. Such $f$ cannot decay too rapidly, by virtue of the following result of N. Levinson.

THEOREM A. *Let $f \in L^1(\boldsymbol{R})$, and suppose for some $\delta > 0$*

$$\int_0^\infty |f(t)| e^{\delta t}\, dt < \infty. \tag{1}$$

*Then, if $\hat{f}(x)$ vanishes throughout any interval, it vanishes identically.*

For the proof, one need only check [**4**, p. 74] that (1) implies that $\hat{f}(x)$ is the boundary value of a function holomorphic in a strip above the real axis. (Actually Levinson, *loc. cit.*, proves much deeper results, with (1) replaced by weaker hypotheses that do not force analyticity of $\hat{f}(x)$. An account of these, based on a new and simple method, will be given by me in a subsequent paper. The weaker Theorem A will serve as a basis for the present discussion.)

Theorem A admits a straightforward generalization to $\boldsymbol{R}^n$. Let us say that a convex cone $K$ in $\boldsymbol{R}^n$ (all cones will be supposed to have vertex at the origin) is *minor* if there exists a unit vector $t^0 \in \boldsymbol{R}^n$ such that $\inf(t^0, t)$; $t \in K, |t| = 1$, is positive. Thus, a half-line in $\boldsymbol{R}^1$, or a sector of opening

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less than $\pi$ in $\boldsymbol{R}^2$, are minor. (It is easy to see that an open convex cone $K$ is minor if and only if there exists a nonsingular linear transformation of $\boldsymbol{R}^n$ carrying $K$ onto the "first quadrant", i.e., the set $K^+$ of points of $\boldsymbol{R}^n$ having all coordinates positive.) By an analyticity argument, as above, one proves easily

THEOREM A′. *Let $f$ be a tempered function on $\boldsymbol{R}^n$ and suppose for some minor cone $K$ and $\delta > 0$*

$$\int_{\boldsymbol{R}^n \setminus K} |f(t)| e^{\delta |t|}\, dt < \infty.$$

*Then, if $f$ has a spectral gap, $f = 0$.*

The condition that $K$ be minor is essential, since for $n \geqq 2$ there exist nontrivial $f \in L^1(\boldsymbol{R}^n)$ which vanish on a half-space and have a spectral gap [**9**, p. 172].

If $f \in L^1(\boldsymbol{R}^n)$ does not vanish identically, any $\phi \in L^\infty(\boldsymbol{R}^n)$ satisfying the convolution equation $f * \phi = 0$ has a spectral gap. Hence it is easy, by duality, to deduce from Theorem A′ the following approximation theorem, as observed recently for $n = 1$ by D. J. Newman [**6**]:

*Let $K$ be a minor cone in $\boldsymbol{R}^n$, $\delta > 0$, and $w(t)$ a nonnegative measurable function on $\boldsymbol{R}^n$ equal to 1 on $K$, and satisfying*

$$\int_{\boldsymbol{R}^n \setminus K} w(t) e^{\delta |t|}\, dt < \infty.$$

*Then, for any $f \in L^1(\boldsymbol{R}^n)$ not identically zero, the translates of $f$ span $L^1(w\, dt)$.*

*In particular, the translates of $f$, restricted to $K$, span the integrable functions on $K$.*

2. Logan, in a 1965 dissertation [**5**, p. 26, Theorem 5.2.1] proved

THEOREM B. *Let $f \in L^\infty(\boldsymbol{R})$ be nonnegative on $\boldsymbol{R}^+$. Then, if $f$ has a spectral gap containing $0$, $f$ vanishes identically.*

Observe that here (and in the next section) the *position* of the spectral gap (i.e. containing the origin) is essential. We sketch a proof, based on a new idea which suggests the correct generalization to $\boldsymbol{R}^n$. We may assume $f \in L^1(\boldsymbol{R})$ (for to reduce the general case to this, consider $f(t) \cdot (\sin \varepsilon t)^2/t^2$ with sufficiently small $\varepsilon$), and that $\hat{f}(t) = 0$ for $|x| \leqq 3$. A simple application of Parseval's formula gives for $m = 0, 1, \ldots$

$$2\pi \int_0^\infty f(t) t^m e^{-t}\, dt = m! \int_{-\infty}^\infty \hat{f}(x)(1 - ix)^{-m-1}\, dx.$$

The integral on the right is bounded by $(\int_{|x| \geqq 3} |x|^{-m-1} dx) \cdot \|\hat{f}\|_\infty = O(3^{-m})$, hence

$$\int_0^\infty f(t)e^t\,dt = \int_0^\infty f(t)\left(\sum_{m=0}^\infty (2t)^m/m!\right) e^{-t}\,dt < C \sum_{m=0}^\infty (2/3)^m < \infty.$$

Now Theorem A implies $f \equiv 0$. Q.E.D.

Let $K, K'$ be closed cones in $\boldsymbol{R}^n$; we say $K'$ is *strongly enclosed* by $K$ if $\{x \in K' : |x| = 1\}$ is in the interior of $K$. We now state our first main result:

THEOREM B′. *Let $f$ be a tempered function on $\boldsymbol{R}^n$ having a spectral gap containing* 0, *and nonnegative a.e. on the closed convex cone K. Let K′ be any closed cone strongly enclosed by K. Then, for some $\delta = \delta(f; K') > 0$, $\int_K f(t)e^{\delta|t|}\,dt < \infty$.*

COROLLARY. *In the hypotheses of Theorem* B′, *if the cone complementary to K is minor, then f vanishes identically.*

The proof of Theorem B′ is in principle like that sketched for Theorem B, but complicated technically. The Corollary then follows using Theorem A′.

REMARK. The hypothesis of nonnegativity on $K$ can be weakened to having range in a sector of opening less than $\pi$.

3. Logan (*loc. cit.*) also established a relation between a spectral gap about 0 and exponential decay of the Poisson integral [**5**, Theorems 6.2.3 and 6.3.1]:

THEOREM C. *Let $f \in L^\infty(\boldsymbol{R})$. The spectrum of f is disjoint from $(-a, a)$ if and only if the Poisson integral*

$$u(x; y) = \frac{1}{\pi}\int_{-\infty}^\infty f(\xi)\frac{y}{(x-\xi)^2 + y^2}\,d\xi$$

*of f satisfies*

$$|u(x; y)| \leqq Ae^{-ay}, \qquad y > 0, \tag{2}$$

*where A is a positive constant independent of x.*

This theorem readily implies that of Paley and Wiener. Logan's proof uses analytic functions. I gave [**8**, p. 152] a proof using only Fourier analysis, which can be extended to $n$ dimensions. Letting $B(x^0; a)$ denote the open ball in $\boldsymbol{R}^n$ with center $x^0$ and radius $a$, we have our second main result:

THEOREM C′. *Let f be a locally integrable function on $\boldsymbol{R}^n$ such that*

$$\int (1 + |x|^2)^{-(n+1)/2}\,|f(x)|\,dx < \infty. \tag{3}$$

*The spectrum of f is disjoint from $B(0; a)$ if and only if the Poisson integral*

$$u(x;y) = c_n \int_{\mathbf{R}^n} f(\xi) \frac{y}{(|x-\xi|^2 + y^2)^{(n+1)/2}} d\xi$$

*of $f$ satisfies, for every $\varepsilon > 0$,*

(4) $$\int |u(x,y)|(1+|x|^2)^{-(n+1)/2} dx \leqq A(\varepsilon)e^{-(a-\varepsilon)y}, \qquad y > 0,$$

*where $A(\varepsilon)$ is a positive number independent of $x$. If* (3) *is replaced by the stronger condition $f \in L^\infty$,* (4) *is to be replaced by*

(4a) $$|u(x,y)| \leqq A(\varepsilon)e^{-(a-\varepsilon)y}, \qquad y > 0.$$

The proof requires estimates for "minimal extrapolations" from the interior, as well as the exterior, of a ball; these will be given elsewhere. As in the case $n = 1$, the "only if" part of the theorem can be strengthened when $f \in L^\infty$.

If, for $k \in L^1(\mathbf{R}^n)$, we denote by $k_{(y)}$ the "dilated function":

$$k_{(y)}(x) = y^{-n}k(y^{-1}x); \qquad x \in \mathbf{R}^n, y > 0,$$

then Theorem C′ (in the case $f \in L^\infty$) may be written: *Let*

(5) $$k(t) = c_n(1+|t|^2)^{-(n+1)/2}$$

*(so that $\hat{k}(x) = e^{-|x|}$); the condition*

(6) $$|(f * k_{(y)})(x)| \leqq A(\varepsilon)\hat{k}((a-\varepsilon)y), \qquad y > 0,$$

*holds for all $\varepsilon > 0$, if and only if the spectrum of $f$ is disjoint from $B(0;a)$.*

Now, this proposition can be established for a large class of kernels $k(t)$ in place of (5), using exactly the same method; *in particular, for* $k(t) = e^{-|t|^2}$, a result obtained otherwise by Ehrenpreis and Malliavin; see [**3**, Corollary 5]. With this special choice of $k$, we may permit $f$ in (6) to be any tempered distribution.

4. Assuming (4a) holds for a *single value* of $x$, we can nonetheless obtain spectral information about $f$. First, some notation: a locally integrable function on $\mathbf{R}^n$ is *anti-radial* if its integral over $B(0;r)$ vanishes for every $r > 0$. Every locally integrable function admits an essentially unique decomposition into a radial and an anti-radial part (for $n = 1$, this is just the even–odd decomposition). We now state our third main result:

THEOREM D. *Let $f$ be a bounded measurable function on $\mathbf{R}^n$ whose Poisson integral $u(x;y)$ satisfies $|u(0;y)| \leqq Ae^{-ay}$. Then, the radial part of $f$ has spectrum disjoint from $B(0;a)$.*

Combining Theorems C′ and D we deduce a proposition solely about harmonic functions: *If $u$ is the Poisson integral of a radial function in*

$L^\infty(\boldsymbol{R}^n)$, (4a) *holds for every* $x \in \boldsymbol{R}^n$ *if it holds for* $x = 0$. For $n = 1$ this is a consequence of a classical Phragmén-Lindelöf theorem, but for $n > 1$ it appears to be new. Another corollary of Theorem D is: *If $u$ is a bounded harmonic function on* $\boldsymbol{R}^n \times \boldsymbol{R}^+$ *satisfying* $u(0; y) = O(e^{-ay})$ *as* $y \to +\infty$, *for every* $a > 0$, *then* $u(0, y)$ *vanishes identically* (*or, what is the same thing,* $u(x; 0+)$ *is an anti-radial function on* $\boldsymbol{R}^n$).

Theorem D also yields a particularly simple proof of the existence of "lacunae" for the wave equation (cf. [**3**, p. 417] for terminology). Also, *the analog of Theorem* D *for the kernel* $k(t) = e^{-|t|^2}$ *is valid*; this is a refinement of a theorem in [**3**].

Proof of Theorem D. Assume first $f \in L^1(\boldsymbol{R}^n)$. We may assume $f$ radial, since the anti-radial part contributes nothing to $u(0; y)$. By assumption,

$$\left| \int_{\boldsymbol{R}^n} f(x) \cdot y(|x|^2 + y^2)^{-(n+1)/2}\, dx \right| \leqq Ce^{-ay}.$$

Substituting here

$$y(|x|^2 + y^2)^{-(n+1)/2} = A_n \int e^{-y|t|} \cdot e^{-i(t,x)}\, dt$$

(where $A_n$ depends only on $n$), and applying Fubini's Theorem, yields (integrations are over $\boldsymbol{R}^n$):

$$A_n \left| \int \hat{f}(t) \cdot e^{-y|t|}\, dt \right| \leqq Ce^{-ay}.$$

Writing $\hat{f}(t) = \phi(|t|)$, we have

$$\int_0^\infty s^{n-1}\phi(s)e^{-ys}\, ds = O(e^{-ay}), \qquad y \to +\infty.$$

Now a simple argument shows that $\phi(s)$ must vanish for $s < a$, hence $\hat{f}(t) = 0$ for $|t| < a$, as we wished to show.

The general case, when $f$ need not belong to $L^1$, leads to serious complications; to be able to perform the crucial "Fubini" step, we replace the Fourier kernel $e^{-i(t,x)}$ by that of Bochner [**2**, p. 112, (5)], in an $n$-dimensional version, and then suitably extend the spectral analysis of Pollard [**7**]. (This is also applicable to $f$ which satisfy only (3).)

5. Let $G$ denote any l.c.a. group, $\hat{G}$ its dual. Let $E$, $\hat{E}$ be closed subsets of $G$, $\hat{G}$ respectively. We say the pair $(E, \hat{E})$ is *interpolatory* if, for every $f \in L^1(G)$, there exists $f_0 \in L^1(G)$ supported in $E$ such that $\hat{f}_0(x) = \hat{f}(x)$, $x \in \hat{E}$. This is equivalent to saying that every function in $L^1(G \backslash E)$ extends to an element of $L^1(G)$ whose Fourier transform vanishes on $\hat{E}$. For instance, one can show when $G = \boldsymbol{R}^n$, that *this is the case if* $\boldsymbol{R}^n \backslash E$ *and* $\hat{E}$ *are compact*. On the other hand, Theorem B′ implies that $(E, \hat{E})$ is not inter-

polatory if $\boldsymbol{R}^n \backslash E$ contains a nonvoid open cone and $\hat{E}$ has interior. Thus, the Fourier transforms of functions supported on proper subcones of $\boldsymbol{R}^n$ are constrained in their local behavior, they possess "local structure". The detailed nature of this local structure was somewhat clarified in [**9**] for the analogous situation in $L^\infty(\boldsymbol{R}^n)$. For $n = 1$ one can show that the presence of arbitrarily long intervals in the complement of the spectrum already forces local structure.

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