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PERIODIC AND HOMOGENEOUS STATES ON A VON NEUMANN ALGEBRA. I[1]

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This paper is devoted to announcing a structure theorem for von Neumann algebras admitting a periodic homogeneous faithful state (see Definitions 1 and 2).

Let $\mathscr{M}$ be a von Neumann algebra. Suppose that ϕ is a faithful normal state on $\mathscr{M}$. We denote by σ_t^ϕ the modular automorphism group of $\mathscr{M}$ associated with ϕ. Let $G(\phi)$ denote the group of all automorphisms of $\mathscr{M}$ which leave ϕ invariant. We introduce the following terminologies concerning ϕ.

DEFINITION 1. If there exists $T > 0$ such that σ_T^ϕ is the identity automorphism of $\mathscr{M}$, denoted by ι, then we call ϕ *periodic*. The smallest such number T is called the *period* of ϕ.

DEFINITION 2. We call ϕ *homogeneous* if $G(\phi)$ acts ergodically on $\mathscr{M}$; that is, the fixed points of $G(\phi)$ are only scalar multiples of the identity.

DEFINITION 3. We call ϕ *ergodic* if $\{\sigma_t^\phi\}$ is ergodic.

The ergodicity of ϕ implies the homogeneity of ϕ, since $\{\sigma_t^\phi\}$ is contained in $G(\phi)$. Furthermore, if $\mathscr{M}$ admits an ergodic state, then $\mathscr{M}$ must be a factor.

Now, suppose ϕ is a periodic homogeneous faithful normal state on $\mathscr{M}$, which will be fixed throughout the discussion. Considering the cyclic representation of $\mathscr{M}$ induced by ϕ, we assume that $\mathscr{M}$ acts on a Hilbert space $\mathfrak{H}$ with a distinguished cyclic vector ξ_0 such that $\phi(x) = (x\xi_0|\xi_0)$, $x \in \mathscr{M}$. According to the theory of modular Hilbert algebras (which the author proposes to call Tomita algebras), there exists the positive self-adjoint operator Δ on $\mathfrak{H}$ and the unitary involution J on $\mathfrak{H}$ such that

$$\sigma_t^\phi(x) = \Delta^{it} x \Delta^{-it}, \qquad x \in \mathscr{M};$$

$$\Delta^{it}\xi_0 = \xi_0;$$

$$J\mathscr{M}J = \mathscr{M}'; \qquad J\Delta^{it}J = \Delta^{it}.$$

Put $\alpha = e^{-2\pi/T}$ with T the period of ϕ. Obviously, we have $0 < \alpha < 1$. We introduce the following notations:

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$$\mathcal{M}_n = \{x \in \mathcal{M} : \sigma_t^\phi(x) = \alpha^{int}x, t \in \mathbf{R}\},$$

$$\mathfrak{H}_n = \{\xi \in \mathfrak{H} : \Delta^{it}\xi = \alpha^{int}\xi, t \in \mathbf{R}\},$$

for $n = 0, \pm 1, \pm 2, \ldots$. Then $\mathcal{M}_0$ is nothing but the centralizer $\mathcal{M}_\phi$ of ϕ in the sense of [**11**, Definition 8.6]. The ergodicity of $G(\phi)$ implies that $\mathcal{M}_n \neq \{0\}$ for every integer n. The subspace $\mathcal{M}_n$ of $\mathcal{M}$ is also given by

$$\mathcal{M}_n = \{x \in \mathcal{M} : \phi(xy) = \alpha^n\phi(yx) \text{ for every } y \in \mathcal{M}\},$$

due to Stϕrmer [**9**].

LEMMA 4. *We have the following*:
(i) $\mathcal{M}_n\mathcal{M}_m \subset \mathcal{M}_{n+m}, \mathcal{M}_n^* = \mathcal{M}_{-n}$;
(ii) $\mathcal{M}_n\mathfrak{H}_m \subset \mathfrak{H}_{n+m}, J\mathfrak{H}_n = \mathfrak{H}_{-n}$;
(iii) $\mathfrak{H} = \sum_{n=-\infty}^{\infty\oplus}\mathfrak{H}_n$;
(iv) $\mathfrak{H}_n = [\mathcal{M}_n\xi_0]$.

It is easily seen that the algebraic direct sum $\sum_{n=-\infty}^{\infty}\mathcal{M}_n$ is a σ-weakly dense *-subalgebra of $\mathcal{M}$. If $\mathcal{N}$ is a von Neumann subalgebra of $\mathcal{M}$ invariant under σ_t^ϕ, then the algebraic direct sum $\sum_{n=-\infty}^{\infty}(\mathcal{N} \cap \mathcal{M}_n)$ is also a σ-weakly dense *-subalgebra of $\mathcal{N}$. Since $\mathcal{M}_n^*\mathcal{M}_n \subset \mathcal{M}_0$ and $\mathcal{M}_n\mathcal{M}_n^* \subset \mathcal{M}_0$, the absolute value $|x|$ of every element x in $\mathcal{M}_n$ falls in $\mathcal{M}_0$. Hence, if $x \in \mathcal{M}_n$ commutes with $\mathcal{M}_0$, then x commutes with x^*x and xx^*, so that x is normal, that is, $x^*x = xx^*$. But this is impossible unless x is in $\mathcal{M}_0$ because $\alpha^n\phi(x^*x) = \phi(xx^*)$. Thus we obtain the following:

PROPOSITION 5. *The relative commutant $\mathcal{M}_0' \cap \mathcal{M}$ of $\mathcal{M}_0$ in $\mathcal{M}$ is contained in $\mathcal{M}_0$ as the center of $\mathcal{M}_0$, denoted by $\mathcal{Z}_0$.*

We denote by π_n the normal representation of $\mathcal{M}_0$ on $\mathfrak{H}_n$ defined by restricting the action of $\mathcal{M}_0$ to $\mathfrak{H}_n$. We also define the antirepresentation π_n' of $\mathcal{M}_0$ on $\mathfrak{H}_n$ by

$$\pi_n'(a) = J\pi_{-n}(a)^*J, \qquad a \in \mathcal{M}_0.$$

For each $x \in \mathcal{M}_n$, we have

$$\pi_n(a)x\xi_0 = ax\xi_0;$$

$$\pi_n'(a)x\xi_0 = xa\xi_0, \qquad x \in \mathcal{M}_0.$$

Hence π_n and π_n' commute. Making use of the ergodicity of $G(\phi)$, we can prove the following:

LEMMA 6. *Both π_n and π_n' are faithful.*

For each $g \in G(\phi)$, we define a unitary operator $U(g)$ on $\mathfrak{H}$ by

$$U(g)x\xi_0 = g(x)\xi_0, \qquad x \in \mathcal{M}.$$

Then the map $:g \in G(\phi) \mapsto U(g)$ is a representation of $G(\phi)$ and covariant with the action of $\mathcal{M}$. It is easily seen that

$$U(g)\pi_n(x)U(g)^* = \pi_n \circ g(x);$$
$$U(g)\pi'_n(x)U(g)^* = \pi'_n \circ g(x), \qquad x \in \mathcal{M}_0, g \in G(\phi).$$

The ergodicity of $G(\phi)$ on $\mathcal{M}_0$ yields that the coupling operator of $\{\pi_n(\mathcal{M}_0), \mathfrak{H}_n\}$ in the sense of Griffin [6] is a scalar multiple of the identity. Therefore, $\{\pi_n(\mathcal{M}_0), \mathfrak{H}_n\}$ has either a separating vector or a cyclic vector.

LEMMA 7. *For $n \geqq 1$, $\{\pi_n, \mathfrak{H}_n\}$ does not have a separating vector.*

PROOF. Since every $\xi \in \mathfrak{H}_n$ is analytic for Δ^{it}, there exists a closed operator a affiliated with $\mathcal{M}$ such that $\xi = a\xi_0$. We can choose a so that $\Delta^{it}a\Delta^{-it} = \alpha^{int}a$. Let $a = uh$ be the polar decomposition of a. Then h is affiliated with $\mathcal{M}_0$ and $u \in \mathcal{M}_n$. If ξ is separating, then $x\xi = 0, x \in \mathcal{M}_0$, implies $x = 0$, so that $xu = 0$ implies $x = 0$. Hence $uu^* = 1$. But $\alpha^n\phi(u^*u) = \phi(uu^*) = 1$, so that $\phi(u^*u) = \alpha^{-n} > 1$ if $n \geqq 1$, a contradiction.

Therefore, $\{\pi_n, \mathfrak{H}_n\}, n \geqq 1$, has a cyclic vector ξ, which is separating for $\pi'_{-n}(\mathcal{M}_0)$. If $a = ku$ is the right polar decomposition of the above a in Lemma 7, then $ux = 0$, $x \in \mathcal{M}_0$, implies $x = 0$, so that we have $u^*u = 1$, and $\phi(uu^*) = \alpha^n$. We choose an element u_1 in $\mathcal{M}_1$ with $u_1^*u_1 = 1$, and fix it. Then u_1^n falls in $\mathcal{M}_n$ for $n \geqq 1$, and $\mathcal{M}_n = \mathcal{M}_0 u_1^n$ because $\mathcal{M}_n u_1^{*n} \subset \mathcal{M}_0$. Therefore we have

$$\mathcal{M}_n = \mathcal{M}_0 u_1^n;$$
$$\mathcal{M}_{-n} = u_1^{*n}\mathcal{M}_0, \qquad n = 1, 2, \ldots .$$

Thus the von Neumann algebra $\mathcal{M}$ is generated by $\mathcal{M}_0$ and the isometry u_1. The choice of u_1 is unique in the following sense:

LEMMA 8. *Every partial isometry v in $\mathcal{M}_1$ is of the form wu_1 with a partial isometry w in $\mathcal{M}_0$.*

Let e_{-n} denote the projections $u_1^n u_1^{n*}$ in $\mathcal{M}_0$ for $n \geqq 1$. Then Lemma 8 implies, together with the ergodicity of $G(\phi)$, that

$$e_{-n}^{\natural} = \alpha^n 1.$$

Thus we conclude that $\mathcal{M}_0$ is of type II_1. We denote by e_n the projection $Je_{-n}J$ in $\mathcal{M}'_0$. Let $\mathfrak{K}_n = e_n\mathfrak{H}_0$, for every integer n.

Define an isomorphism θ of $\mathcal{M}_0$ onto $e_{-1}\mathcal{M}_0 e_{-1}$ by $\theta(x) = u_1 x u_1^*$, $x \in \mathcal{M}_0$. Then the isomorphism θ induces an automorphism $\tilde{\theta}$ of $\mathcal{Z}_0$ by the equality $\theta(a) = \tilde{\theta}(a)e_{-1}, a \in \mathcal{Z}_0$. It follows from Lemma 8 that $\tilde{\theta}$ does not depend on the choice of u_1.

PROPOSITION 9. *The center $\mathcal{Z}$ of $\mathcal{M}$ is precisely the fixed point subalgebra of $\mathcal{Z}_0$ with respect to $\tilde{\theta}$. Therefore, $\mathcal{M}$ is a factor if and only if $\tilde{\theta}$ is ergodic on $\mathcal{Z}_0$.*

PROPOSITION 10. *For $n \geqq 1$, we have*

$$\{\pi_n, \mathfrak{H}_n\} \cong \{\pi_0, \mathfrak{K}_n\};$$

$$\{\pi_{-n}, \mathfrak{H}_{-n}\} \cong \{\theta^n, \mathfrak{K}_{-n}\},$$

where $\{\pi_0, \mathfrak{K}_n\}$ means the restriction of π_0 to the invariant subspace $\mathfrak{K}_n$.

We denote by ϕ_0 the restriction of ϕ to $\mathcal{M}_0$.

THEOREM 11. *In the pre-Hilbert space metric given by the state ϕ, the von Neumann algebra $\mathcal{M}$ is decomposed as*

$$\mathcal{M} = \cdots \oplus u_1^{*n}\mathcal{M}_0 \oplus \cdots \oplus u_1^*\mathcal{M}_0 \oplus \mathcal{M}_0 \oplus \mathcal{M}_0 u_1 \oplus \cdots \oplus \mathcal{M}_0 u_1^n \oplus \cdots.$$

The algebraic structure of $(\mathcal{M}, \phi)$ is determined by $\{\mathcal{M}_0, \theta, \phi_0\}$ in the following sense: Let $\overline{\mathcal{M}}$ be another von Neumann algebra equipped with a periodic homogeneous faithful state $\bar{\phi}$ of period T and let $\overline{\mathcal{M}}$ be decomposed with respect to $\bar{\phi}$ as

$$\overline{\mathcal{M}} = \cdots \oplus \bar{u}_1^*\overline{\mathcal{M}}_0 \oplus \cdots \oplus \bar{u}_1^*\overline{\mathcal{M}}_0 \oplus \overline{\mathcal{M}}_0 \oplus \overline{\mathcal{M}}_0 \bar{u}_1 \oplus \cdots \oplus \overline{\mathcal{M}}_0 \bar{u}_1^n \oplus \cdots.$$

Suppose $\bar{u}_1$ gives rise to an isomorphism of $\bar{\theta}$ of $\overline{\mathcal{M}}_0$ onto $\bar{e}_{-1}\mathcal{M}_0\bar{e}_{-1}$. Then there exists an isomorphism σ of $\mathcal{M}$ onto $\overline{\mathcal{M}}$ with $\phi = \bar{\phi} \circ \sigma$ if and only if there exists an isomorphism σ_0 of $\mathcal{M}_0$ onto $\overline{\mathcal{M}}_0$ and a partial isometry w in $\mathcal{M}_0$ such that $w\theta(x)w^* = \sigma_0^{-1} \circ \bar{\theta} \circ \sigma_0(x), x \in \mathcal{M}_0$, and $\phi_0 = \bar{\phi}_0 \circ \sigma$, where ϕ_0 (resp. $\bar{\phi}_0$) means the restriction of ϕ (resp. $\bar{\phi}$) to $\mathcal{M}_0$ (resp. $\overline{\mathcal{M}}_0$).

Conversely, if $\mathcal{M}_0$ is a von Neumann algebra of type II_1. Let e be a projection of $\mathcal{M}_0$ with $e^\natural = \alpha$, $0 < \alpha < 1$. Suppose θ is an isomorphism of $\mathcal{M}_0$ onto $e\mathcal{M}_0 e$. Then θ induces an automorphism $\tilde{\theta}$ of the center $\mathcal{Z}_0$ of $\mathcal{M}_0$ such that $\tilde{\theta}(a)e = \theta(a), a \in \mathcal{Z}_0$. Let ϕ_0 be a $\tilde{\theta}$-invariant faithful normal state on $\mathcal{Z}_0$. We extend ϕ_0 to a faithful normal trace on $\mathcal{M}_0$ by $\phi_0(x) = \phi_0(x^\natural), x \in \mathcal{M}_0$. Suppose G denotes the group of all automorphisms g of $\mathcal{M}_0$ such that there exists a partial isometry w_g in $\mathcal{M}_0$ with $g \circ \theta \circ g^{-1}(x) = w_g\theta(x)w_g^*$, and such that $\phi_0 \circ g = \phi_0$ (this is satisfied automatically if $\tilde{\theta}$ is ergodic). Such an automorphism is called *admissible*.

THEOREM 12. *In the above situation, if G acts ergodically on the center $\mathcal{Z}_0$, then there exists a von Neumann algebra $\mathcal{M}$ with a periodic homogeneous faithful state ϕ of period $T = -2\pi/\log \alpha$ such that $\{\mathcal{M}_0, \theta, \phi_0\}$ appears in the decomposition of $\mathcal{M}$ associated with ϕ as described in Theorem 11.*

We denote by $\mathcal{R}(\mathcal{M}_0, \theta, \phi_0)$ the von Neumann algebra determined by $(\mathcal{M}_0, \theta, \phi_0)$ in Theorems 11 and 12. We can describe the automorphism group $G(\phi)$ in terms of G and the unitary group of $\mathcal{Z}_0$. In order to distinguish the algebraic type of $\mathfrak{R}(\mathcal{M}_0, \theta, \phi_0)$, we employ new results of A. Connes [**4**] concerning modular automorphism groups.

For a von Neumann algebra $\mathscr{M}$, let Aut($\mathscr{M}$) (resp. Int($\mathscr{M}$)) denote the group of all (resp. inner) automorphisms of $\mathscr{M}$. Let Out($\mathscr{M}$) denote the quotient group Aut($\mathscr{M}$)/Int($\mathscr{M}$). A. Connes showed recently that the canonical image $\dot{\sigma}_t^\phi$ of the modular automorphism group σ_t^ϕ in Out($\mathscr{M}$) does not depend on the choice of ϕ; hence we denote it simply by $\dot{\sigma}_t$. Furthermore he proved that if σ_T^ϕ is inner for some $T > 0$, then σ_T^ϕ is given by a unitary operator in the center of the centralizer $\mathscr{M}_\phi$ of ϕ.

Now, we return to the original situation. In order to avoid any possible confusion, we denote by T_0 the period of our state ϕ.

THEOREM 13. *For $T > 0$, σ_T^ϕ is inner, that is, $\dot{\sigma}_T =$* identity, *if and only if α^{-iT} is a point spectrum of the automorphism $\tilde{\theta}$ of $\mathscr{Z}_0$.*

Therefore, if we have ergodic automorphisms $\tilde{\theta}$ in $\mathscr{Z}_0$ of different point spectral type, then the resulting factors $\mathscr{R}(\mathscr{M}_0, \theta, \phi_0)$ are nonisomorphic.

EXAMPLES. Let $\mathscr{F}$ denote a hyperfinite II_1-factor and $\mathscr{A} = L^\infty(0, 1)$. Let $\mathscr{M}_0 = \mathscr{F} \otimes \mathscr{A}$. For $0 < \alpha < 1$, we choose a projection $f \in \mathscr{F}$ with $\tau(f) = \alpha$, where τ is the canonical trace of $\mathscr{F}$. It is then known that there exists an isomorphism θ_1 of $\mathscr{F}$ onto $f\mathscr{F}f$. Let $\tilde{\theta}$ be an ergodic automorphism of $\mathscr{A}$ with invariant faithful normal state μ. Let $\theta_0 = \theta_1 \otimes \tilde{\theta}$ and $\phi_0 = \tau \otimes \mu$. Then the triplet $\{\mathscr{M}_0, \theta, \phi_0\}$ satisfies all our requirements, since the automorphism id $\otimes\, \tilde{\theta}^n$, $n = 0 \pm 1, \pm 2, \ldots$, are admissible and ergodic on the center $\mathscr{Z}_0 = 1 \otimes \mathscr{A}$. Thus, if we choose various kinds of ergodic automorphisms $\tilde{\theta}$, then we get different kinds of modular groups $\dot{\sigma}_t$ as well as different factors.

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